Mètodes Numèrics . Eng. Química , UPC

Curs 2007-8. Tema 7: Problemes resolts.

1 (i) $y(0)=2$, $y'(0)=-1$ (cond inicials)

$$y''(0)=2\cdot 0\cdot y'(0)-y(0)=-2$$

$$y'''=2y'+2xy''-y'=y'+2xy''$$

$$y'''(0)=-1+2\cdot 0\cdot y''(0)=-1$$

$$y^{4)}=y''+2\cancel{x}y''+2xy''' , \quad y^{4)}(0)=3y''(0)=-6$$

$$y(x)=2-x+\frac{(-2)}{2!}x^2+\frac{(-1)}{3!}x^3+\frac{(-6)}{4!}x^4+\theta(x^5)$$

$$=2-x-x^2-\frac{x^3}{6}-\frac{x^4}{4}+\theta(x^5)$$

L'edo no té punts singulars $\Rightarrow$ el radi de convergència és $\rho=\infty$

(ii) Solució: $y(x)=\frac{x^3}{6}+\theta(x^5)$

Radi de convergència infinit.

(iii) $y'=-\frac{y}{x^2}$ $\qquad y''=-\frac{y'}{x^2}+2\frac{y}{x^3}$

$$y'''=-\frac{y''}{x^2}+2\frac{y'}{x^3}+2\frac{y'}{x^3}-6\frac{y}{x^4}=-\frac{y''}{x^2}+4\frac{y'}{x^3}-6\frac{y}{x^4}$$

$$y^{4)}=-\frac{y'''}{x^2}+2\frac{y''}{x^3}+4\frac{y''}{x^3}-12\frac{y'}{x^4}-6\frac{y'}{x^4}+24\frac{y}{x^5}$$

D'on $y'(1) = -\frac{y(1)}{1^2} = 2$

$$y''(1) = -\frac{y'(1)}{1^2} + 2\frac{y(1)}{1^3} = -2-4 = -6$$

$$y'''(1) = -\frac{y''(1)}{1^2} + 4\frac{y'(1)}{1^3} - 6\frac{y(1)}{1^4} = 6+8+12 = 26$$

$$y^{4)}(1) = -26-36-36-48 = -146$$

$$y(x) = -2 + 2(x-1) - 3(x-1)^2 + \frac{13}{3}(x-1)^3 - \frac{73}{12}(x-1)^4 + O(x-1)^5$$

$x=0$ és singular $\Rightarrow$ radi de convergència $\rho = |1-0| = 1$.

(2) (i) Tots els punts són regulars

(ii) $y'' = \frac{y'}{\cos 2x} + \frac{2x}{\cos 2x}$

Pts singulars: $\cos 2x = 0 \Leftrightarrow 2x = \frac{\pi}{2} + k\pi,\ k\in\mathbb{Z}$

$$\Updownarrow$$

$$x = \pi/4 + k\frac{\pi}{2},\ k\in\mathbb{Z}$$

(iii) $y' = \frac{x}{\sin(x)}\, y$

Pts singulars: $\sin x = 0 \Leftrightarrow x = k\pi,\ k\in\mathbb{Z}$,

però $\lim_{x\to 0}\frac{x}{\sin x} = 1$

$\boxed{x = k\pi,\ k\in\mathbb{Z}, k\neq 0}$

7 (i) $\begin{cases} y' = (2x-1)y + \sin(x) \\ y(0) = -2 \end{cases}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[(2x-1)y + \sin x\right] = 2x-1$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(0,-2) = -1 < 0 \Rightarrow \text{PVI estable}$$

(ii) $\frac{\partial}{\partial y}\left(y/x + x^2\right) = 1/x$

$$\frac{\partial f}{\partial y}(\underset{x_0}{\underset{\downarrow}{0.1}}, \underset{y_0}{\underset{\downarrow}{-0.5}}) = \frac{1}{0.1} = 10 > 0 \Rightarrow \text{PVI inestable}$$

8 (i) $y_1(x) = y(x)$, $y_2(x) = y'(x)$

$y_1'(x) = y_2$

$y_2'(x) = y'' = -y = -y_1$

Sistema: $\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

A té VAPs $\lambda = i, -i$ amb $Re\lambda = 0 \Rightarrow$ no podem dir res de l'estabilitat.

(ii) $\frac{dy}{dt} = \underbrace{\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}}_{A} y$

A té VAPs $\lambda = 6 > 0$, $\lambda = -2 < 0$

$\Rightarrow$ solucions inestables.

(iii) $y_1(x) = y(x)$, $y_2(x) = y'(x)$

$y_1' = y' = y_2$

$y_2' = y'' = \underbrace{1 + x + (x-1)^2 - (x-1)^2 y_1 + \sqrt{1-x}\, y_2}_{F(x,\vec{y})}$

$\begin{pmatrix} y_1(0.5) \\ y_2(0.5) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (la condició inicial)

$D_y F = \begin{pmatrix} 0 & 1 \\ -(x-1)^2 & \sqrt{1-x} \end{pmatrix}$

$D_y F(\underset{x_0}{\underset{\shortparallel}{0.5}}, \underbrace{2, 1}_{\vec{y}(x_0)}) = \begin{pmatrix} 0 & 1 \\ -1/4 & 1/\sqrt{2} \end{pmatrix}$

VAPs: $\lambda = \underset{\underset{0}{\vee}}{\frac{1}{2\sqrt{2}}} \pm \frac{1}{2\sqrt{2}} i \quad \Rightarrow$ PVI inestable

9 (i) $y' = \underbrace{-(x+2)\,y + g(x)}_{f(x,y)}$

$$\frac{\partial f}{\partial y} = -(x+2)$$

Per $x \in [1,3]$, $\frac{\partial f}{\partial y} = -(x+2) \in [-5,-3]$

Volem h tal que $\frac{\partial f}{\partial y} \cdot h > -2$

$\Rightarrow h < 0.4$

(ii) $f(x,y) = e^{-x} y + x y^2$

$$\frac{\partial f}{\partial y} = \underset{0}{\overset{\vee}{e^{-x}}} + 2\underset{0}{\overset{\vee}{x}}\,y$$

per $x \in [1,3]$

Signe de $y(x)$? $y(1) = 13 > 0$

$y' = e^{-x} y + x y^2 > 0$ mentre $\begin{cases} x > 0 & \text{(sempre per } x \in [1,3]) \\ y > 0 \end{cases}$

$\Rightarrow$ $y(x)$ creixent mentre $y(x) > 0$

Com $y(1) > 0$, $y(x)$ creixent i positiva

$\Rightarrow \frac{\partial f}{\partial y} > 0 \Rightarrow$ no hi ha rigidesa

(hi ha inestabilitat, que és pitjor!)

(iii) Convertim en sistema d'ordre 1

$(y_1(x) = y(x),\ y_2(x) = y'(x))$:

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} y_2 \\ x^2 - \dfrac{48\, y_2}{10 - y_1} \end{pmatrix}}_{F(x,\, y_1,\, y_2)}, \quad \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$D_y F = \begin{pmatrix} 0 & 1 \\ \dfrac{48\, y_2}{(10 - y_1)^2} \cdot (-1) & \dfrac{-48}{10 - y_1} \end{pmatrix}$$

Discutim rigidesa només en $x = 1$:

$$D_y F(1, 4, 6) = \begin{pmatrix} 0 & 1 \\ -8 & -8 \end{pmatrix}$$

VAPs: $\lambda = -4 + 2\sqrt{2} \simeq -1.17,\ -4 - 2\sqrt{2} \simeq -6.83$

PVI estable i rígid. Cal triar pas h amb $(-4 - 2\sqrt{2})h > -2 \iff \underline{h < \dfrac{2}{6.83}} \simeq \underline{0.29}$

14 (i) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

↳ té VAPs $4, 2 > 0$

$\Rightarrow$ inestable

(ii) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ té VAPs $-1 < 0$, $2 > 0 \Rightarrow$ inestable

(iii) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ té VAPs $0, 1 > 0$ $\Rightarrow$ inestable

(iv) $\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.7 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.25 \\ 0.2 & 0.2 & 0.45 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

té VAP $1 > 0 \Rightarrow$ inestable

(v) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

té VAPs $\lambda = -1+3i, -1-3i$ amb $\mathrm{Re}\,\lambda < 0$

$\Rightarrow$ estable

15 (i) $\underbrace{\begin{cases} \dot{x} = -x \\ \dot{y} = 1 - x^2 - y^2 \end{cases}}_{}$ $F(x,y)$

Punts d'equilibri $\Leftrightarrow$ $\begin{cases} -x = 0 \\ 1 - x^2 - y^2 = 0 \end{cases}$

Solucions: $\begin{cases} x = 0 \\ 1 - y^2 = 0 \Rightarrow y = 1 \text{ o } y = -1 \end{cases}$

Dos punts d'equilibri: $\begin{cases} P_1 = (0,1) \\ P_{-1} = (0,-1) \end{cases}$

Estabilitat:

$$DF(x,y) = \begin{pmatrix} -1 & 0 \\ -2x & -2y \end{pmatrix}$$

$$DF(P_1) = DF(0,1) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$ VAPs $-1, -2 < 0$ $\Rightarrow$ estable

$$DF(P_{-1}) = DF(0,-1) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$ VAPs $-1 < 0$, $2 > 0 \Rightarrow$ inestable

(ii) $\begin{cases} \dot{x} = y \\ \dot{y} = -2\sin x - 2y \end{cases}$

Punts d'equilibri:

$$\begin{cases} y=0 \\ -2\sin x - 2y = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ \sin x = 0 \Rightarrow x = k\cdot\pi, \quad k\in\mathbb{Z} \end{cases}$$

són $P_0=(0,0)$, $P_1=(\pi,0)$, $P_2=(2\pi,0)$, ...

$P_{-1}=(-\pi,0)$, $P_{-2}=(-2\pi,0)$, ...

Estabilitat:

$$DF(x,y) = \begin{pmatrix} 0 & 1 \\ -2\cos x & -2 \end{pmatrix}$$

$$DF(P_k) = \begin{pmatrix} 0 & 1 \\ -2\cos(k\pi) & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\cdot 2 & -2 \end{pmatrix}$$

Cas k parell: $DF(P_k) = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$ té VAPs $-1\pm i$ $\Downarrow$ P_k estables

Cas k senar:

$DF(P_k) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}$ té VAPs $\lambda_1 \simeq 0.73$, $\lambda_2 \simeq -2.73$

$\lambda_1 > 0$ $\Rightarrow$ inestables.

```
%Problema 16
%Emprant el solve es troben els punts d'equilibri:
x=solve('-10*u+10*v','28*u-v-u*w','-(8/3)*w+u*w');
x.u
x.v
x.w
%resultats:
%x.u=
%[   0]
%[ 8/3]

% x.v =
%[   0]
%[ 8/3]

%x.w=
%[  0]
%[ 27]
%dos punts d'equilibiri:(0,0,0) i (8/3,8/3,27)

% Estabilitat i rigidesa dels punts critics:
u=0;v=0;w=0;
J=[-10,10,0;28-w,-1,-u;w,0,-8/3+u]
eig(J)
%  surten -22.8277   11.8277   -2.6667
% per tant es inestable

u=8/3;v=8/3;w=27;
J=[-10,10,0;28-w,-1,-u;w,0,-8/3+u]
eig(J)
% surten -14.4488   1.7244 + 6.8453i   1.7244 - 6.8453i
% per tant es inestable.


%per al punt(0,0,0);
 [x15s,y15s]=ode15s(@fun16,[0 10],[0.1 0.1 0.1]);

%l'ode 45 tarda massa estona, degut al VAP <<0
% de la Jacobiana (rigidesa)

% la funcio a integrar amb ode15s/ode45 es

function du=fun16(t,u)
du(1,1)=-10*u(1)+10*u(2);
du(2,1)=28*u(1)-u(2)-u(1)*u(3);
du(3,1)=-(8/3)*u(3)+u(1)*u(3);
```