# SNA in Harper maps: a laboratory of Strange Nonchaotic Attractors.

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# Motivation

- In dissipative systems, the occurence of strange attractors (SA) has been detected and there is nowadays a mathematical background for such systems.
- In many physically relevant situations, one has to consider systems with mixed dynamics: dissipative for some variables side and quasi-periodic for some others, e.g. skew-product systems.
- Since the early 1980's, there are experimental studies on the occurence of Strange Nonchaotic Attractors (SNA) in such systems: the attractor is geometrically complicated, orbits are (mostly) attracted to this set and dynamics on the set is nonchaotic.
- In contrast with the mathematical aparatus for SA, that for SNA seems to be much more limited. . .
- However using a convenient "dictionary", examples of SNA in some families can be derived from previous mathematical knowledge.

# Harper map

Family of Harper maps

$$
\begin{aligned}
y_{n+1} &= \underbrace{\frac{1}{a - b\cos(2\pi\theta_n) - y_n}}_{f_{a,b}(y_n,\theta_n)}, \\
\theta_{n+1} &= \theta_n + \omega \qquad (\text{mod } 1),
\end{aligned}
$$

- $y \in \overline{\mathbb{R}} = [-\infty, +\infty]$ and $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $b$ (coupling), $a$ (energy) and $\omega$ (irrational frequency) are parameters.

A Harper map is a skew-product map on $\overline{\mathbb{R}} \times \mathbb{T}$

$$F_{a,b,\omega}(y_n, \theta_n) = (f_{a,b}(y_n, \theta_n), \theta_n + \omega),$$

$$F^{(N)}_{a,b,\omega}(y_0, \theta_0) = (f^{(N)}_{a,b}(y_0, \theta_0), \theta_0 + N\omega), \quad N \in \mathbb{Z}.$$

# Harper map in polar coordinates

To get rid of the point at $\infty$, take polar coordinates $\varphi = \arctan y \in \mathbb{P} \simeq [-\pi/2, \pi/2]$,

$$\begin{aligned} \varphi_{n+1} &= \underbrace{\arctan\left(\frac{1}{a - b\cos(2\pi\theta_n) - \tan\varphi_n}\right)}_{\tilde{f}_{a,b}(\varphi_n,\theta_n)}, \\ \theta_{n+1} &= \theta_n + \omega \qquad (\mathrm{mod}\ 1). \end{aligned}$$

which is also a skew-product map on $\mathbb{P} \times \mathbb{T}$

$$\tilde{F}_{a,b,\omega}(\varphi_n, \theta_n) = (\tilde{f}_{a,b}(\varphi_n, \theta_n), \theta_n + \omega),$$

$$\tilde{F}^{(N)}_{a,b,\omega}(\varphi_0, \theta_0) = (\tilde{f}^{(N)}_{a,b}(\varphi_0, \theta_0), \theta_0 + N\omega), \quad N \in \mathbb{Z}.$$

# Harper maps & Operators

After writing $y_n = x_{n-1}/x_n$ we obtain a *Harper equation*,

$$x_{n+1} + x_{n-1} + b\cos(2\pi(\theta_0 + n\omega))\,x_n = a x_n, \qquad n \in \mathbb{Z}$$

which are the eigenvalue equation of a Schrödinger operator:

$$\left(H_{b,\omega,\theta_0} x\right)_n = x_{n+1} + x_{n-1} + b\cos(2\pi(\theta_0 + n\omega))\,x_n .$$

known as Harper or Almost Mathieu operator.

- It is a bounded and self-adjoint operators on $l^2(\mathbb{Z})$.
- If $\omega$ is irrational, the spectrum is independent of $\theta_0$.
- There is a solid theory for these operators.

# The Harper Linear skew-product

The first-order system associated to Harper equation is the Harper linear skew-product:

$$\begin{aligned}
\underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{v_{n+1}} &= \underbrace{\begin{pmatrix} a - b\cos(2\pi\theta_n) & -1 \\ 1 & 0 \end{pmatrix}}_{M_{a,b}(\theta_n)} \underbrace{\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}}_{v_n}, \\
\theta_{n+1} &= \theta_n + \omega \pmod 1,
\end{aligned}$$

whose evolution is given by the Harper cocycle

$$M^{(N)}_{a,b,\omega}(\theta_0) = \begin{cases} M_{a,b}(\theta_{N-1}) \dots M_{a,b}(\theta_0) & \text{if } N > 0, \\ Id & \text{if } N = 0, \\ M^{-1}_{a,b}(\theta_N) \dots M^{-1}_{a,b}(\theta_{-1}) & \text{if } N < 0. \end{cases}$$

In its study, Mather / Sacker-Sell spectral theory and Floquet theory for linear skew-products are very useful tools.

# Lyapunov exponents

For any $v_0 \in \mathbb{R}^2 \setminus \{0\}$, the Lyapunov exponent of the skew-product

$$\lambda_{a,b,\omega}(v_0, \theta_0) = \lim_{N \to +\infty} \frac{1}{N} \log |v_N| = \lim_{N \to +\infty} \frac{1}{N} \log \left| M^{(N)}_{a,b,\omega}(\theta_0) v_0 \right|$$

and if $v_0 = (x_{-1}, x_0)$ and $\varphi_0 = \arctan(x_{-1}/x_0)$, the Lyapunov exponent of the Harper map

$$\beta_{a,b,\omega}(\varphi_0, \theta_0) = \lim_{N \to +\infty} \frac{1}{N} \log \left| \frac{\partial \varphi_N}{\partial \varphi_0}(\varphi_0, \theta_0) \right| = \lim_{N \to +\infty} \frac{1}{N} \log \left| m^{(N)}_{a,b,\omega}(\varphi_0, \theta_0) \right|$$

$$m^{(N)}_{a,b,\omega}(\varphi_0, \theta_0) = \frac{\partial \tilde{f}_{a,b}}{\partial \varphi}(\varphi_{N-1}, \theta_{N-1}) \cdots \frac{\partial \tilde{f}_{a,b}}{\partial \varphi}(\varphi_0, \theta_0) \ .$$

An easy computation shows the relation

$$\beta_{a,b,\omega}(\varphi_0, \theta_0) = -2\lambda_{a,b,\omega}(v_0, \theta_0).$$

# Oseledec theory

Oseledec (1968): for a.e. initial condition $(v_0, \theta_0)$ the Lyapunov exponent exists and equals the averaged Lyapunov exponent.

$$\lambda_{a,b,\omega}(v_0, \theta_0) = \overline{\lambda}_{a,b,\omega} = \lim_{N \to +\infty} \frac{1}{N} \int_{\mathbb{T}} \log \left| M^{(N)}_{a,b,\omega}(\theta) \right| d\theta \geq 0$$

If $\overline{\lambda}_{a,b,\omega} > 0$ (hyperbolic case), there is a full measure set $\Theta \subset \mathbb{T}$ s.t.

$$\mathbb{R}^2 = W^s(\theta) \oplus W^u(\theta), \quad \theta \in \Theta. \tag{1}$$

The invariant subbundles $W^{s/u} := \left\{ (W^{s/u}(\theta), \theta), \theta \in \Theta \right\}$ are defined by

$$\text{(stable subbundle)} \quad v \in W^s(\theta) \Leftrightarrow \lim_{N \to \pm\infty} \frac{1}{N} \log \left| M^{(N)}(\theta) v \right| = -\overline{\lambda}$$

$$\text{(stable subbundle)} \quad v \in W^u(\theta) \Leftrightarrow \lim_{N \to \pm\infty} \frac{1}{N} \log \left| M^{(N)}(\theta) v \right| = +\overline{\lambda}.$$

# Uniform vs. Nonuniform hyperbolicity

A key point is the regularity of the splitting in case of positive Lyapunov exponents.

The following is equivalent in the hyperbolic case:

- The only bounded solution of the linear skew-product $(v_n, \theta_n)_n$ is the trivial one, i.e. $v \equiv 0$;
- $W^s$ and $W^u$ are defined for all $\theta$, i.e. $\Theta = \mathbb{T}$;
- $W^s$ and $W^u$ depend continuously on $\theta$.
- $W^s$ and $W^u$ depend analytically on $\theta$.

(Sacker-Sell, Selgrade, Johnson . . . ).

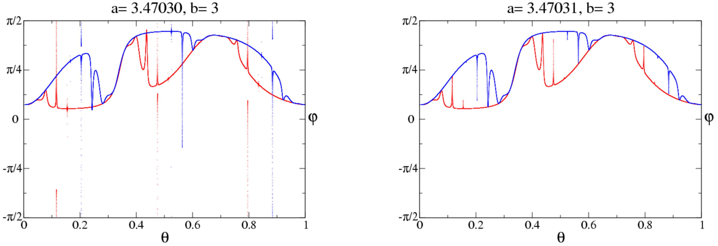
If any of these conditions hold, we speak of uniform hyperbolicity and otherwise of nonuniform hyperbolicity

Idea: Nonuniform hyperbolicity of the Harper cocycle $\Leftrightarrow$ SNA in the Harper map.

Define $\varphi^s(\theta)$ and $\varphi^u(\theta)$ as the “angles” of $W^s(\theta)$ and $W^u(\theta)$. Then the graphs

$$\Phi^s = \{(\varphi^s(\theta),\theta),\ \theta\in\Theta\} \text{ and } \Phi^u = \{(\varphi^u(\theta),\theta),\ \theta\in\Theta\}$$

are invariant under the Harper map and have quasi-periodic dynamics.




$\rightarrow$ Thus $\Phi^s$ and $\Phi^u$ are nonchaotic invariant sets.

# From an unstable subbundle to an attracting curve

In the hyperbolic case (uniform or not), for $\theta \in \Theta$ (i.e. for a.e. $\theta$),

$$\beta(\varphi, \theta) = \lim_{N \to +\infty} \frac{1}{N} \log \left| \tilde{f}^{(N)}_{a,b,\omega}(\varphi, \theta) - \varphi^u(\theta_N) \right|_{\mathbb{P}} = -2\overline{\lambda} < 0,$$

thus $\Phi^u$ attracts exponentially almost every orbit on $\mathbb{P} \times \mathbb{T}$.

$\to \Phi^u$ is a nonchaotic attractor.

Equivalently, $\Phi^s$ attracts exponentially almost every orbit at $N \to -\infty$.

$\to \Phi^s$ is a nonchaotic repellor.

# From nonuniformly hyperbolicity to SNA

- If the Harper skew-product is uniformly hyperbolic, then $\Phi^u$ (resp. $\Phi^s$) are analytic attracting (resp. repelling) invariant curves.
- If the skew-product is nonuniformly hyperbolic, then the functions $\varphi^u$, $\varphi^s$ are measurable but not continuous functions of $\theta$.
- By the quasi-periodic dynamics, $\varphi^u$,$\varphi^s$ are nowhere continuous.

To sum up in the nonuniformly hyperbolic case, we will say that $\Phi^u$ is a Strange Nonchaotic Attractor (SNA) because:

(i) $\Phi^u$ is the graph of a measurable function of $\theta$, $\varphi^u$, which is nowhere continuous ($\Phi^u$ is Strange);

(ii) $\Phi^u$ is an invariant set of the Harper map with quasi-periodic dynamics ($\Phi^u$ is Nonchaotic);

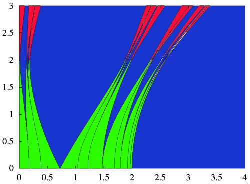
(iii) Almost every orbit in phase space is attracted to $\Phi^u$ at exponential rate ($\Phi^u$ is an Attractor).

# Nonuniform Hyperbolicity in the Harper skew-product

- $a$ is in the spectrum of the Harper operator $\Leftrightarrow$ the skew-product does not have an exponential dichotomy (Johnson).
- For the Lyapunov exponent, Herman proved the bound

$$\overline{\lambda}_{a,b,\omega} \geq \max\left(0, \log \frac{|b|}{2}\right)$$

with equality $\Leftrightarrow$ $a$ is in the spectrum (Bourgain & Jitomirskaya).
- Therefore the Harper skew-product is nonuniformly hyperbolic $\Leftrightarrow$ $a$ is in the spectrum and $|b| > 2$.
- A Harper map has a SNA $\Leftrightarrow$ $a$ is in the spectrum and $|b| > 2$.



# Example

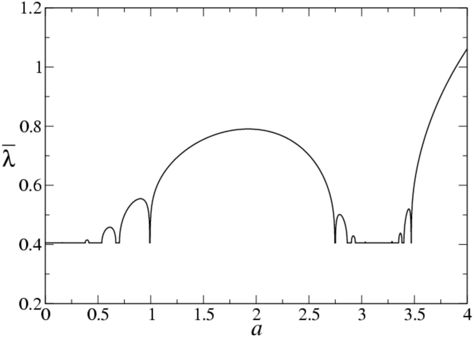



Figure: The maximal Lyapunov exponent $\bar{\lambda}$ as a function of $a$. Here $b = 3$ and $\omega = e/4$.

# Different gaps, different indices

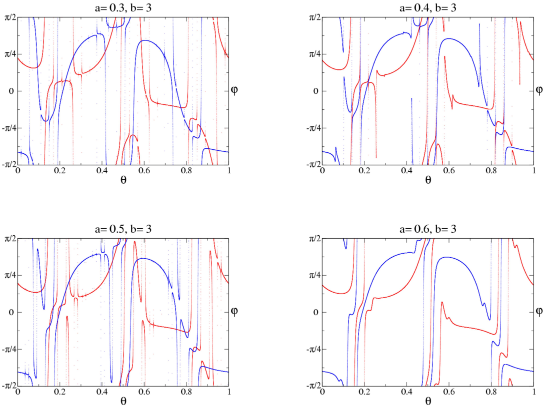


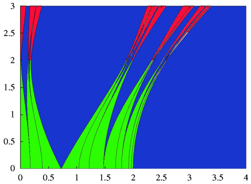

Figure: The attractor $\Phi^u$ and the repellor $\Phi^s$ of the Harper map. $a = 0.4$ and $a = 0.6$ are different gaps labelled by the indices 8 and 5, respectively.

# Persistence of SNA: Topology vs. Measure

- The spectrum is a Cantor set & inside the gaps there are analytic attracting invariant curves $\Rightarrow$ SNA are not persistent by perturbations of $a$.
- However, when $|b| > 2$ the spectrum is a large set in measure:

$$\text{measSpec}\left(H_{b,\omega,\phi}\right) = |4 - 2|b||,$$

  so many SNA persist.
- Moving $a$ and $b$ some persistent curves of SNA can be created. These are analytic for large $|b|$.
- The most trivial is $a = 0$, which has a SNA when $|b| > 2$.



# Collision and creation of SNA

- Inside a gap $\varphi^u$ and $\varphi^s$ are analytic.
- As *a* approaches the endpoint of a gap, they become closer and closer.
- Since the endpoint of a gap belongs to the spectrum, the distance must go to zero as *a* converges to it.

# Example

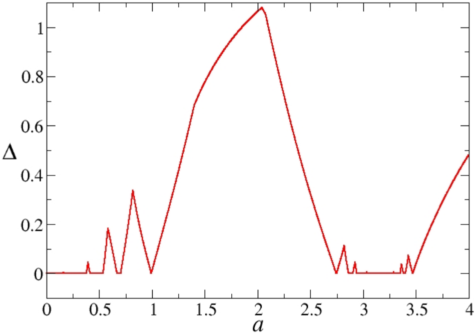



Figure: Minimal distance $\Delta$ between $\Phi^u$ and $\Phi^s$ as a function of $a$. Here $b = 3$ and $\omega = e/4$.

# Localized eigenfunctions and *SNA*

- Solutions of the eigenvalue equation which decay exponentially in $|n|$ can be interpreted as heteroclinic solutions which connect $\Phi^u$ (when $n \to -\infty$) and $\Phi^s$ (when $n \to +\infty$).
- If $a$ is a point eigenvalue at the endpoint of a gap, duality shows that that it is formed when $\theta = k\omega$.