Cantor Spectrum for Quasi-Periodic Schrödinger Operators

Joaquim Puig
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona

The Almost Mathieu operator

The Almost Mathieu operator:

$$(H^{AM}_{b,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \qquad (1)$$

on $l^2(\mathbb{Z})$, where

- b is a real parameter,
- ω is an irrational number and
- $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

$H^{AM}_{b,\omega,\phi} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a quasi-periodic Schrödinger operator (Q-P S.O.) which is bounded and self-adjoint. The eigenvalue equation of the Almost Mathieu operator is

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = a x_n, \qquad n \in \mathbb{Z},$$

sometimes called Harper's equation, which is a discretization of the Mathieu equation

$$x'' + (a + b\cos(t))\,x = 0.$$

studied by Ince (1922).

IDS & The spectrum of Q-P S.O.

Let H_ϕ be a Q-P. S.O. (for example $H_\phi = H^{AM}_{b,\omega,\phi}$),

$$k_L(a, H_\phi) = \frac{1}{(L-1)} \# \left\{ \text{eigenvalues} \le a \text{ of } H_\phi|_{\{1,\ldots,L-1\}} \right\}$$

with zero boundary conditions at both ends. Then

$$\lim_{L\to\infty} k_L(a, H_\phi) = k_{H_\phi}(a),$$

the integrated density of states (IDS) exists. Moreover

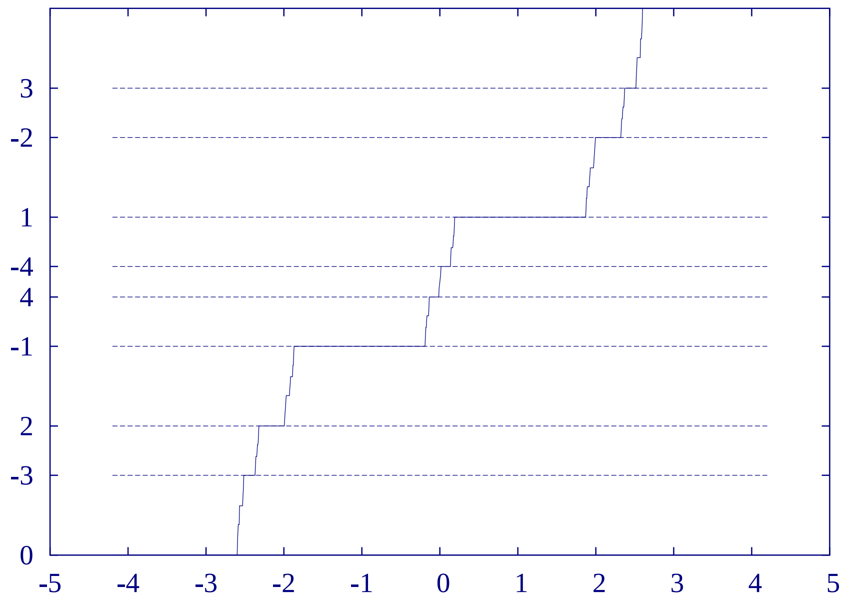
- It is independent of ϕ.
- Continuously increasing on a.
- $a \in \text{Spec}(H_\phi)$ $\Leftrightarrow$ the IDS is increasing at a.
- In particular $\text{Spec}(H_\phi)$ independent of ϕ.

Gap labelling & Collapse of Gaps

Johnson & Moser (1982): if $a \notin \mathrm{Spec}(H_\phi)$,

$$k(a, H_\phi) = \{n\omega\}, \quad (\{\cdot\} \text{ denotes fractional part})$$

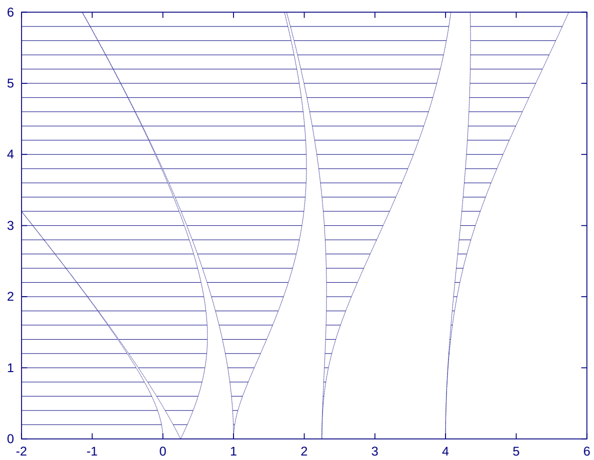
for some $n \in \mathbb{Z}$. Gap labelling for $H^{AM}_{2,\omega,\phi}$ and $b = 2$:



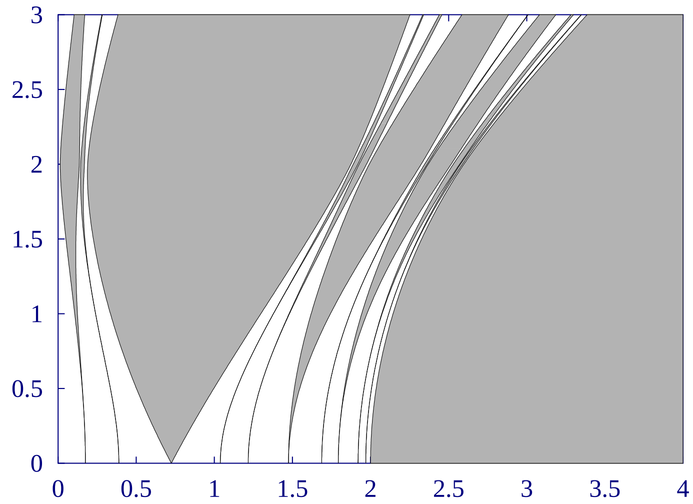
Notation. Let $n \in \mathbb{Z}$:

- $k^{-1}(\{n\omega\})) = [a_-, a_+] \Rightarrow (a_-, a_+)$ is called a noncollapsed or open gap.
- $k^{-1}(\{n\omega\})) = \{a_\pm\} \Rightarrow \{a_\pm\}$ is called a collapsed gap.

Mathieu vs. Almost Mathieu



Gaps in Mathieu operator



Gaps in Almost Mathieu

Key: If ω irrational, $\{n\omega\}, n \in \mathbb{Z}$ is dense in $[0,1]$, and if all gaps are noncollapsed (or these are dense) the spectrum of a Q-P S.O. is a Cantor set.

Kac & Simon (1981) Let ω irrational & $b \neq 0$, prove that the spectrum of the Almost Mathieu Operator, $\sigma^{AM}_{b,\omega}$,

- is a Cantor set (Ten Martini Problem).
- has all gaps open (Dry Ten Martini Problem).

We want to prove:

Corollary 1 (CMP. V. 244, N. 2) *Assume that $\omega \in \mathbb{R}$ is Diophantine, that is, there exist positive constants c and $r > 1$ such that*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

for all $n \neq 0$. Then, the spectrum of the Almost Mathieu operator, $\sigma_{b,\omega}^{AM}$, is a Cantor set if $b \neq 0, \pm 2$.

Remark 2 *Very recently Avila & Jitomirskaya have proved Cantor structure for all irrational frequencies.*

In the Diophantine case, Cantor structure is an (almost) direct consequence of

Theorem 3 (Jitomirskaya(1999)) *Let ω be Diophantine. Then, if $|b| > 2$ the operator $H_{b,\omega,0}^{AM}$ has only pure point spectrum with exponentially decaying eigenfunctions.*

About the Dry version, a consequence of Eliasson (1992) is

Corollary 4 *Let $\omega \in \mathbb{R}$ be Diophantine. Then, there is a constant $C = C(\omega) > 0$ such that if $0 < |b| < C$ or $4/C < |b| < \infty$ all the spectral gaps of $\sigma_{b,\omega}^{AM}$ are open.*

Starting point

Let $b > 2$ and ω Diophantine

Jitomirskaya proves that, if $b > 2$, then $H^{AM}_{b,\omega,0}$ has pure-point spectrum with exponentially decaying eigenfunctions.

What do we need from this result?

There exists a dense subset in $\sigma^{AM}_{b,\omega}$ of point eigenvalues of $H^{AM}_{b,\omega,0}$ whose eigenvectors are exponentially localized.

Why is this enough?

Let a be one of this eigenvalues and $\psi = (\psi_n)_{n\in\mathbb{Z}}$ its exponentially localized eigenvector. We are going to see that a is the endpoint of a noncollapsed spectral gap.

Aubry duality

By hypothesis we have $a \in \sigma_{b,\omega}^{AM}$ and $\psi \in l^2(\mathbb{Z})$ which satisfy the Harper equation

$$\psi_{n+1} + \psi_{n-1} + b\cos(2\pi\omega n)\psi_n = a\psi_n, \qquad n \in \mathbb{Z},$$

with some constants $A, \beta > 0$ such that

$$|\psi_n| \le A\exp(-\beta|n|), \qquad n \in \mathbb{Z} \qquad \text{and} \qquad \psi \not\equiv 0.$$

By Aubry duality, the Fourier transform of ψ,

$$\tilde{\psi}(\theta) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}, \qquad \theta \in \mathbb{T}$$

is real analytic in $|\text{Im } \theta| < \beta$ and the quasi-periodic Bloch wave

$$x_n = \tilde{\psi}\left(2\pi\omega n + \theta\right), \qquad n \in \mathbb{Z},$$

for any $\theta \in \mathbb{T}$, satisfies the equation

$$(x_{n+1} + x_{n-1}) + \frac{4}{b}\cos(2\pi\omega n + \theta)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z},$$

which is again a Harper equation

$$(a, b) \mapsto \left(\frac{2a}{b}, \frac{4}{b}\right).$$

The dual system

By Avron & Simon,

$$k^{AM}_{b,\omega}(a) = k^{AM}_{4/b,\omega}\left(\frac{2a}{b}\right)$$

if we prove so that if we prove that $\frac{2a}{b}$ is the endpoint of a noncollapsed gap of $\sigma^{AM}_{4/b,\omega}$ we are done.

Writing the first-order system, we have that $(x_n)_{n\in\mathbb{Z}}$ satisfies

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

with $\theta_0 = \theta$. In terms of $\tilde{\psi}$,

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega+\theta) \\ \tilde{\psi}(2\pi\omega+\theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega+\theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

holds for all $\theta \in \mathbb{T}$.

The Dynamical Approach

The eigenvalue equation of a Q-P S.O. (e.g. Harper's equation):

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n = a x_n, \qquad n \in \mathbb{Z}.$$

can be written as a quasi-periodic skew-product on $\mathbb{R}^2 \times \mathbb{T}$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix}}_{A_{a-V}(\theta_n)} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega.$$

which is the iteration of a quasi-periodic cocycle on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$(x,\theta) \in \mathbb{R}^2 \times \mathbb{T} \mapsto (A_{a-V}(\theta_n), \omega))\,(x,\theta) = (A_{a-V}(\theta_n)x, \theta + 2\pi\omega)$$

that is,

$$x_{n+1} = \begin{pmatrix} a - V(2\pi\omega n + \phi) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a - V(\phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot x_0$$

and

$$\theta_n = 2\pi\omega n + \theta_0.$$

Conjugation and Reducibility of Cocycles

Two cocycles (A,ω) and (B,ω) are conjugated if there exists a continuous $Z:\mathbb{T}\to SL(2,\mathbb{R})$ such that

$$A(\theta)Z(\theta)=Z(\theta+2\pi\omega)B(\theta), \qquad \theta\in\mathbb{T}.$$

In this case the skew-products

$u_{n+1}=A(\theta)u_n, \qquad \theta_{n+1}=\theta_n+2\pi\omega$ and $v_{n+1}=B(\theta)v_n, \qquad \theta_{n+1}=\theta_n+2\pi\omega$

are conjugated by means of the change $u=Zv$.

A cocyle (A,ω) is reducible to constant coefficients if it is conjugated to a cocycle (B,ω) with B constant.

Remark 5 *B is called the Floquet matrix. Neither B nor Z are unique.*

The fundamental solution of a reducible system $X_n(\phi)$ has the following Floquet representation:

$$X_n(\phi)=Z(2\pi n\omega+\phi)B^nZ(\phi)^{-1}X_0(\phi).$$

Is our dual skew-product reducible to constant coefficients?

From Bloch waves to reducibility

Recall that for the dual system we have

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega+\theta) \\ \tilde{\psi}(2\pi\omega+\theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega+\theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

In this case one can see the following

Lemma 6 *Let $A:\mathbb{T}\to SL(2,\mathbb{R})$ be a real analytic map, with analytic extension to $|Im\ \theta|<\delta$ and ω be Diophantine. Assume that there is a nonzero real analytic map $v:\mathbb{T}\to\mathbb{R}^2$, with analytic extension to $|Im\ \theta|<\delta$ such that*

$$v(\theta+2\pi\omega)=A(\theta)v(\theta)$$

holds for all $\theta\in\mathbb{T}$. Then, the quasi-periodic skew-product flow given by

$$u_{n+1}=A(\theta_n)u_n, \qquad \theta_{n+1}=\theta_n+2\pi\omega, \tag{2}$$

with $(u_n,\theta_n)\in\mathbb{R}^2\times\mathbb{T}$ for all $n\in\mathbb{Z}$ is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|Im\ \theta|<\delta$. Moreover the Floquet matrix can be chosen to be of the form

$$B=\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{3}$$

for some $c\in\mathbb{R}$.

Endpoints of Gaps and Reducibility

Given a quasi-periodic Schrödinger operator

$$(H_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z},$$

consider the corresponding quasi-periodic Schrödinger cocycle,

$$(A_{a-V}, \omega) = \left(\begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \omega \right), \tag{4}$$

with V real analytic and ω Diophantine (for simplicity). Assume that, for some value of a, (4) is reducible to the constant coefficients cocycle (B, ω). Then:

- a is at the endpoint of a spectral gap of $\sigma(H_{V,\omega,\phi}) \Leftrightarrow$ trace $B = \pm 2$.
- The gap is collapsed $\Leftrightarrow B = \pm I$

Therefore, if

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

then the gap is collapsed $\Leftrightarrow c = 0$.

Ince's argument & Cantor spectrum for the Almost Mathieu Operator

In the Almost Mathieu case...

If $B = I$ $(c = 0)$ there are two linearly independent real analytic quasi-periodic solutions with frequency ω of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos(2\pi\omega n + \phi)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}.$$

Passing to the dual, this means that

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n)x_n = ax_n, \qquad n \in \mathbb{Z}.$$

has two linearly independent solutions in $l^2(\mathbb{Z}) \Rightarrow$ Contradiction!.

Therefore $B \neq I$ $(c \neq 0) \Rightarrow 2a/b$ is the endpoint of a noncollapsed gap of $\sigma^{AM}_{4/b,\omega}$.

Such a's are dense in the spectrum...$\Rightarrow$ noncollapsed spectral gaps are dense in the spectrum.

Idempotent Floquet matrices at endpoints of Noncollapsed gaps

Theorem 7 (Johnson, 1981) *Let ω be irrational en $a_0 \in \rho_{b_0} = \mathbb{R} - \sigma^{AM}_{b,\omega}$ if, and only if, the skew product*

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 - b_0 \cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

has an exponential dichotomy: it is reducible to constant coefficients with a hyperbolic Floquet matrix.

Then

Lemma 8 *If $|\alpha| \neq 0$ is small enough and $c\alpha < 0$ then*

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} + \alpha - \frac{4}{b} \cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

has an exponential dichotomy, and, therefore $2a/b$ is the endpoint of a non-collapsed spectral gap of $\sigma^{AM}_{4/b}$

Remark 9 *This also holds for more general potentials V.*

Extension to real analytic potentials

Let us try to reproduce the proof for more general potentials...

- If $V : \mathbb{T} \to \mathbb{R}$ is real analytic, Bourgain & Jitomirskaya (2002) prove that for some $\varepsilon > 0$, if $|V|_\rho < \varepsilon$ and ω is Diophantine, the long-range operator,

$$\left(L_{V,\omega,\phi}x\right)_n = \sum_{k\in\mathbb{Z}} V_k x_{n+k} + 2\cos(2\pi\omega n + \phi)x_n$$

  has pure-point spectrum with exponentially localized eigenfunctions for almost all $\phi \in \mathbb{T}$.

- Since $L_{V,\omega,\phi}$ is the dual model of $H_{V,\omega,\phi}$, one can show that for Lebesgue almost every $a \in \mathbb{R}$, the cocycle

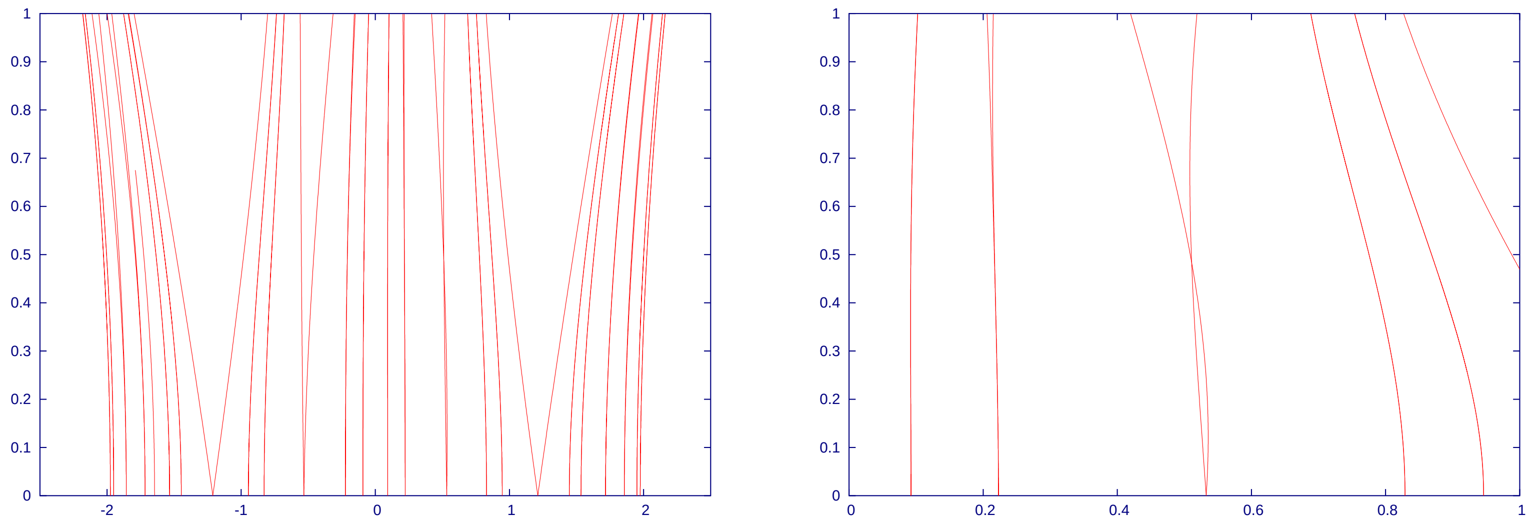
$$(A_{a-V}, \omega) = \left(\begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \omega \right)$$

  is reducible to constant coefficients.

- Also, for V and ω as above, there exists a dense set of a's in the spectrum such that the corresponding cocycle is reducible to

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

For $V(\theta) = \cos(\theta) + 0.3\cos(2\theta)$



- Such $a's$ are endpoints of gaps. However we cannot use Ince's argument to conclude $c \neq 0$. In fact c can be zero and there are examples where the spectrum is not a Cantor set.

- Nevertheless, even if c can be zero, generically it is different from zero and the spectrum is a Cantor set.

- Nonperturbative localization results can be used to generalize results by Moser & Pöschel (1984) and Eliasson (1991) on the genericity of Cantor spectrum (in the reducible setting).

  What about Cantor spectrum in parametric families of operators ?

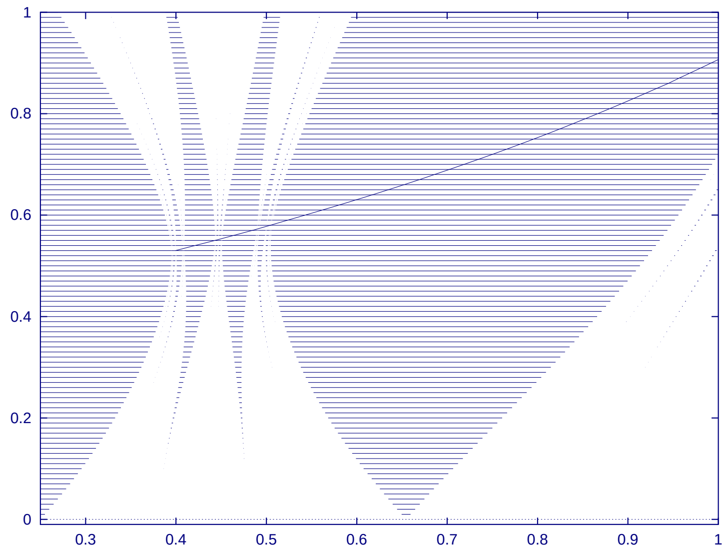
Cantor spectrum for specific models

Let us consider the continuous case (joint with Broer & Simó)...

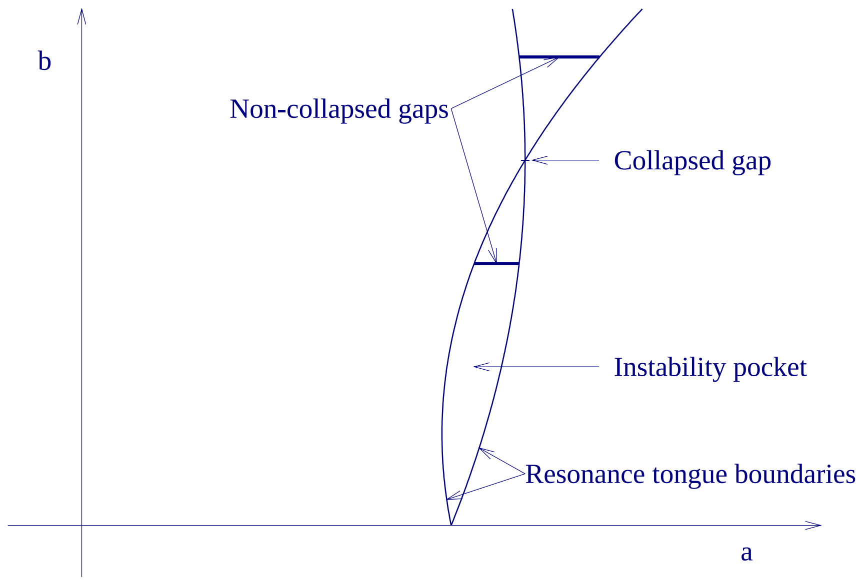
Theorem 10 *Let* $d \geq 2$. *Then for almost all* $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$, *with* $d \geq 2$, *there is a* $C = C(\omega)$ *such that for all values of* b, *except for a countable set, with* $|b| < C$, *the spectrum of the quasi-periodic Mathieu operator*

$$H^{QPM}_{b,\omega,\phi} x = -x'' + b \sum_{j=1}^{d} \cos(\omega_j t) x$$

has all gaps open and, thus, it is a Cantor set.



Strategy: analyticity of gap boundaries



- Using KAM techniques we prove that gaps (either collapsed or noncollapsed) are real analytic functions of b for $|b|$ small.
- Using Birkhoff Normal Form, we show that all these "gap functions" (or resonance tongue boundaries) have some finite order of contact at $b = 0$.
- In particular, each gap can collapse at most a finite number of times.
- The number of gaps is countable and the result follows.