

Nonperturbative Reducibility and Irreducibility

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A interesting class of dynamical systems are **quasi-periodic cocycle** on $SL(2, \mathbb{R})$

$$\begin{aligned} (A, \omega) : \quad SL(2, \mathbb{R}) \times \mathbb{T}^d &\longrightarrow SL(2, \mathbb{R}) \times \mathbb{T}^d \\ (X, \theta) &\longmapsto (A(\theta)X, \theta + 2\pi\omega), \end{aligned}$$

which define, by iteration, a **quasi-periodic skew-product** either on $SL(2, \mathbb{R}) \times \mathbb{T}^d$

$$(X_n, \theta_n) = (A, \omega)^n (X_0, \theta_0)$$

or $\mathbb{R}^2 \times \mathbb{T}^d$,

$$(v_n, \theta_n) = (A, \omega)^n (v_0, \theta_0).$$

Quasi-periodic Schrödinger cocycles

The eigenvalue equation of a q-p. **Schrödinger operator**

$$(H_{V,\omega,\theta}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \theta)x_n, \quad n \in \mathbb{Z}.$$

where $V : \mathbb{T}^d \rightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^d$ is irrational is a difference equation

$$x_{n+1} + x_{n-1} + V(2\pi\omega n + \theta)x_n = a, \quad n \in \mathbb{Z}$$

which generates a quasi-periodic **Schrödinger cocycle** defined by

$$A_{a,V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

Conjugacy & Reducibility of Cocycles

Two cocycles (A, ω) and (B, ω) are **conjugated** if there is a **conjugation** $Z : \mathbb{T}^d \rightarrow SL(2, \mathbb{R})$ s.t.

$$(A, \omega) \circ (Z, 0) = (Z, 0) \circ (B, \omega)$$

or, equivalently

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B(\theta), \quad \theta \in \mathbb{T}.$$

If B is independent of θ , (A, ω) is **reducible to constant coefficients** and B is a **Floquet matrix**.

- The class of conjugacy **depends** on the regularity of the conjugation: $C^0, C^1, C^\infty, C_\rho^a, L^2 \dots$
- **Frequency-halving** is sometimes necessary.

In the complementary of the spectrum, dynamics is **uniformly hyperbolic**:

Theorem (Johnson)

Let V continuous and ω irrational. The following is equivalent:

- a is **not** in the spectrum of $H_{V,\omega,\phi}$
- The cocycle $(A_{a,V}, \omega)$ has an **exponential dichotomy**: it is conjugated to a diagonal cocycle (D, ω) with $|\text{trace } D(\theta)| > 2$.
- (if, e.g., V is real analytic and ω Diophantine), the cocycle $(A_{a,V}, \omega)$ is reducible to **constant coefficients** with **hyperbolic Floquet matrix**.

Reducibility in the spectrum

It requires some regularity ...

Let $C_\rho^a(\mathbb{T}^d, \mathbb{R})$ the space of **real analytic** $V : \mathbb{T}^d \rightarrow \mathbb{R}$ such that

$$|V|_\rho := \sup_{|\operatorname{Im} \theta| < \rho} |V(\theta)| < \infty$$

and $DC^d(c, \tau)$ the **Diophantine frequencies** $\omega \in \mathbb{R}^d$ such that

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{c}{|\mathbf{k}|^\tau}, \quad \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \quad (2)$$

If the **upper Lyapunov exponent**

$$\beta(\mathbf{a}, V, \omega) = \lim_{N \rightarrow \infty} \frac{1}{N+1} \int_{\mathbb{T}} \|A_{\mathbf{a}, V}(2\pi N\omega + \theta) \cdots A_{\mathbf{a}, V}(\theta)\| d\theta \quad (3)$$

is positive in the spectrum, there can be no reducibility. Thus, reducibility in the spectrum needs V “small”.

Eliasson's reducibility theorem

Let $\rho > 0$ and $\omega \in DC(c, \tau)$. Then, Eliasson (1992), among other things, showed that there is a $\varepsilon_E = \varepsilon_E(c, \tau, \rho) > 0$ such that if

$$|V|_\rho < \varepsilon_E$$

- 1 $(A_{a,V}, \omega)$ is analytically **reducible** to constant coefficients for **almost all** $a \in \mathbb{R}$.
- 2 For a **generic** $V \in C_\rho^a(\mathbb{T}^d, \mathbb{R})$, the spectrum $\sigma(V, \omega)$ is a **Cantor set**.
- 3 For a **generic** $V \in C_\rho^a(\mathbb{T}^d, \mathbb{R})$, there exist $a \in \sigma(V, \omega)$ for which $(A_{a,V}, \omega)$ is **not reducible** to constant coefficients.

→ This result is optimal for $d \geq 2$.

→ The set of "reducible energies" can be characterized.

A nonperturbative version in $d = 1$

Let $\rho > 0$. Then, there is a $\varepsilon = \varepsilon(\rho) > 0$ such that if

$$|V|_\rho < \varepsilon$$

and $\omega \in DC(c, \tau)$ then the following holds

- 1 $(A_{a,V}, \omega)$ is **reducible** to constant coefficients for **almost all** $a \in \mathbb{R}$.
 - 2 For a **generic** $V \in C_\rho^a(\mathbb{T}, \mathbb{R})$, the spectrum $\sigma(V, \omega)$ is a **Cantor set**.
 - 3 If $\sigma(V, \omega)$ is a **Cantor set**, there is a residual set in $\sigma(V, \omega)$ for which $(A_{a,V}, \omega)$ is **not reducible** to constant coefficients.
- Therefore nonreducibility is a “byproduct” of Cantor spectrum.

Idea of proof (I): Bloch waves & Aubry duality

- The (analytic) reducibility of a Schrödinger cocycle with Floquet matrix in $SO(2, \mathbb{R})$ is **equivalent** to find a pair of **Bloch waves** for the Harper-like equation

$$\mathbf{x}_n^\pm(\phi) = e^{\pm i\varphi n} \psi(2\pi\omega n + \phi), \quad n \in \mathbb{Z},$$

where $\psi : \mathbb{T} \rightarrow \mathbb{C}$ is analytic.

- The Fourier coefficients of ψ , $(\psi_k)_{k \in \mathbb{Z}}$, satisfy a q.p. difference equation

$$\sum_{k \in \mathbb{Z}} V_k \psi_{n-k} + 2 \cos(2\pi\omega n + \varphi) \psi_n = a \psi_n \quad n \in \mathbb{Z}.$$

which is Harper-like only in the **Almost Mathieu case**

Idea of proof (II) Nonperturbative localization

- Bourgain & Jitomirskaya show that, if $|V|_\rho < \varepsilon$ and ω is Diophantine, the operator

$$(L_{V,\omega,\varphi}\psi)_n = \sum_{k \in \mathbb{Z}} V_k \psi_{n-k} + 2 \cos(2\pi\omega n + \varphi) \psi_n$$

is **pure-point** with **exponentially decaying eigenfunctions** for almost all $\varphi \in \mathbb{T}$.

- Undoing Fourier transform, we obtain that the union of eigenvalues of $L_{V,\omega,\varphi}$, for almost all φ 's is a set of reducible a 's.
- Using Deift & Simon'83, this set has **full Lebesgue measure** on $\sigma(V, \omega)$ because $\beta = 0$ there.

Intervals with $\beta = 0$ are not generic ...

Using the analyticity of m -functions, as in Avila & Jitomirskaya. If

- $\omega \in \mathbb{R}^d$ Diophantine and
- $V : \mathbb{T}^d \rightarrow \mathbb{R}$ is real analytic with $|V|_\rho < \infty$.
- $\sigma(V, \omega)$ contains an open interval where the Lyapunov exponent equals zero

then

- The cocycle $(A_{a,V}, \omega)$ is analytically reducible to constant coefficients for **all** $a \in \sigma(V, \omega)$.
- The Floquet matrix lies in $SO(2, \mathbb{R})$.
- For any $\delta > 0$ there is a **generic** $W \in C_\rho^a(\mathbb{T}^d, \mathbb{R})$ with

$$|V - W|_\rho < \delta$$

such that $\sigma(V, \omega) \cap I$ is a Cantor set.

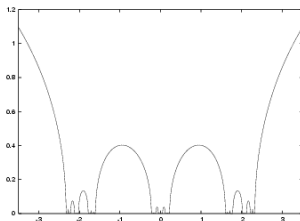
... and in Cantor sets irreducibility is generic!

- $V : \mathbb{T}^d \rightarrow \mathbb{R}$ be **continuous**.
- $\omega \in \mathbb{R}^d$ **rationally independent**.
- There is an interval such that

$$I \cap \sigma(V, \omega) = \{a \in I; \beta(a) = 0\} \neq \emptyset$$

is a **Cantor set**.

Then there is a residual G_δ of $I \cap \sigma(V, \omega)$ whose energies **not** reducible by a continuous transformation.



Idea of proof

- Fact 1: If a_0 is reducible to a $SO(2, \mathbb{R})$ -cocycle, $a \mapsto \beta(a)$ is Lipschitz at a_0
- Fact 2: If a_0 is at the endpoint of a open gap in $I \cap \sigma(V, \omega)$,

$$s_I(a) = \sup_{a \in I \setminus \{a_0\}} \frac{\beta(a) - \beta(a_0)}{a - a_0} = \infty$$

- Fact 3: The Lyapunov exponent is continuous in I .

Therefore, for any $n = 1, 2, \dots$ the sets

$$U(n) = \{a \in I \cap \sigma(V, \omega); s_I(a) > n\}$$

is open and dense. Their intersection is the set of nonreducible energies.

Kotani considered the set where the Lyapunov exponent vanishes

$$A_V^0 = \{a \in \sigma(V, \omega); \beta(a) = 0\}$$

in the more general ergodic case.

- When this set has **positive measure**, he proved **L^2 -reducibility** to a measurable $SO(2, \mathbb{R})$ -valued cocycle. for **almost all** a in A_V^0 .
- When the set A_V^0 contains an interval, the reducibility **extends to the whole interval** (cf. DeConcini & Johnson).
- The previous argument also shows that **if $A_V^0 = \sigma(V, \omega)$ is a Cantor set**, there is also a residual set of energies where this L^2 -reducibility to a rotation does **not** hold.