The Ten Martini Problem

(The Proof)

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The Almost Mathieu operator

The Almost Mathieu operator:

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \tag{1}$$

on $l^2(\mathbb{Z})$, where

- b is a real parameter,
- ω is an irrational number and
- $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

We want to prove:

Corollary 1 (CMP. V. 244, N. 2) *Assume that $\omega \in \mathbb{R}$ is Diophantine, that is, there exist positive constants c and $r > 1$ such that*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

for all $n \neq 0$. Then, the spectrum of the Almost Mathieu operator, σ_b, is a Cantor set if $b \neq 0, \pm 2$.

Starting point

Let $b > 2$ and ω Diophantine

Jitomirskaya proves that, if $b > 2$, then $H_{b,0}$ has pure-point spectrum with exponentially decaying eigenfunctions.

What do we need from this result?

There exists a dense subset in σ_b of point eigenvalues of $H_{b,0}$ whose eigenvectors are exponentially localized.

Why is this enough?

Let a be one of this eigenvalues and $\psi = (\psi_n)_{n\in\mathbb{Z}}$ its exponentially localized eigenvector. We are going to see that a is the endpoint of a non-collapsed spectral gap.

Aubry duality

By hypothesis we have $a \in \sigma_b$ and $\psi \in l^2(\mathbb{Z})$ which satisfy the Harper equation

$$\psi_{n+1} + \psi_{n-1} + b\cos(2\pi\omega n)\psi_n = a\psi_n, \qquad n \in \mathbb{Z},$$

with some constants $A, \beta > 0$ such that

$$|\psi_n| \leq A\exp(-\beta|n|), \qquad n \in \mathbb{Z} \qquad \text{and} \qquad \psi \not\equiv 0.$$

By Aubry duality, the Fourier transform of ψ,

$$\tilde{\psi}(\theta) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}, \qquad \theta \in \mathbb{T}$$

is real analytic in $|\text{Im } \theta| < \beta$ and the sequence

$$x_n = \tilde{\psi}\left(2\pi\omega n + \theta\right), \qquad n \in \mathbb{Z},$$

for any $\theta \in \mathbb{T}$, satisfies the equation

$$(x_{n+1} + x_{n-1}) + \frac{4}{b}\cos(2\pi\omega n + \theta)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}.$$

The dual system

By Avron & Simon (duality of the IDS), if we prove that $\frac{2a}{b}$ is the endpoint of a non-collapsed gap of $\sigma_{4/b}$ we are done.

Writing the first-order system, we have that $(x_n)_{n\in\mathbb{Z}}$ satisfies

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

with $\theta_0 = \theta$. In terms of $\tilde{\psi}$,

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega + \theta) \\ \tilde{\psi}(2\pi\omega + \theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega + \theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

What can one say about the other solutions?

The Dynamical Approach

- The eigenvalue equation of the Almost Mathieu operator (Harper equation):

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = a x_n, \qquad n \in \mathbb{Z}.$$

- The associated first-order system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a - b\cos(\theta_n) & -1 \\ 1 & 0 \end{pmatrix}}_{A(a-b\cos(\theta_n))} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega.$$

- The quasi-periodic skew-product on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$X_{n+1} = A(a - b\cos(\theta_n))X_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega,$$

that is,

$$X_{n+1} = \begin{pmatrix} a - b\cos(2\pi\omega n + \phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a - b\cos(\phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot X_0$$

being X_0 a 2×2 matrix with $\det X_0 = 1$.

Quasi-periodic Skew-Products & Cocycles

A quasi-periodic skew-product flow on $SL(2,\mathbb{R}) \times \mathbb{T}$,

$$X_{n+1} = A(\theta_n) X_n \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{2}$$

with $A : \mathbb{T} \to SL(2,\mathbb{R})$ can be seen as the iteration of the following quasi-periodic cocycle

$$\begin{array}{rccc} (A,\omega): & SL(2,\mathbb{R}) \times \mathbb{T} & \longrightarrow & SL(2,\mathbb{R}) \times \mathbb{T} \\ & (X,\theta) & \mapsto & (A(\theta)X, \theta + 2\pi\omega)\,, \end{array}$$

so that, in this language,

$$(X_n, \theta_n) = (A,\omega)^n (X_0, \theta_0)\,.$$

As an example, we have the Almost Mathieu cocycle, given by

$$A(\theta) = \begin{pmatrix} a - b\cos(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \qquad \theta \in \mathbb{T}$$

or, in general, any Schrödinger cocycle

$$A(\theta) = \begin{pmatrix} a - bV & -1 \\ 1 & 0 \end{pmatrix}, \qquad \theta \in \mathbb{T}$$

where $V : \mathbb{T} \to \mathbb{R}$.

Example: constant cocycles

Assume that (A, ω) is a cocycle with $A(\theta) = A_0 \in SL(2, \mathbb{R})$ for all $\theta \in \mathbb{T}$. Then,

$$(A, \omega)^n = (A_0^n, n\omega)$$

If λ, λ^{-1} are the eigenvalues of A_0, we can classify the dynamics of the skew-product

$$u_{n+1} = A(\theta)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

as follows

- Hyperbolic case: $|\text{trace } A_0| > 2$, solutions are linear combinations of λ^n and λ^{-n} $\Rightarrow$ No bounded solutions
- Elliptic case: $|\text{trace } A_0| < 2$ all solutions are linear combinations of λ^n and λ^{-n} $\Rightarrow$ All solutions are quasi-periodic with frequency $\arg \lambda$.
- Parabolic case: $|\text{trace } A_0| = 2$ and $A_0 \neq \pm I$ $\Rightarrow$ one constant solution and another growing linearly.
- Elliptic degenerate case: $A_0 = \pm I$ $\Rightarrow$ all solutions of (I, ω) or $(-I, \omega)^2$ are constant.

We have a quantitative picture of the solutions.

Conjugation and Reducibility of Cocycles

Two cocycles (A, ω) and (B, ω) are conjugated if there exists a continuous $Z : \mathbb{T} \to SL(2, \mathbb{R})$ such that

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B(\theta), \qquad \theta \in \mathbb{T}.$$

In this case the skew-products

$$u_{n+1} = A(\theta)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \text{ and } v_{n+1} = B(\theta)v_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

are conjugated by means of the change $u = Zv$.

A cocyle (A, ω) is reducible to constant coefficients if it is conjugated to a cocycle (B, ω) with B constant.

Remark 2 *B is called the Floquet matrix. Neither B nor Z are unique.*

The fundamental solution of a reducible system $X_n(\phi)$ has the following Floquet representation:

$$X_n(\phi) = Z(2\pi n\omega + \phi)B^n Z(\phi)^{-1} X_0(\phi).$$

Is our dual skew-product reducible to constant coefficients?

Going back to our case

Recall that for the dual system we have

$$\begin{pmatrix} \tilde{\psi}(4\pi\omega+\theta) \\ \tilde{\psi}(2\pi\omega+\theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}(2\pi\omega+\theta) \\ \tilde{\psi}(\theta) \end{pmatrix}$$

In this case one can see the following

Lemma 3 *Let $A:\mathbb{T}\to SL(2,\mathbb{R})$ be a real analytic map, with analytic extension to $|Im\ \theta|<\delta$ and ω be Diophantine. Assume that there is a nonzero real analytic map $v:\mathbb{T}\to\mathbb{R}^2$, with analytic extension to $|Im\ \theta|<\delta$ such that*

$$v(\theta+2\pi\omega) = A(\theta)v(\theta)$$

holds for all $\theta\in\mathbb{T}$. Then, the quasi-periodic skew-product flow given by

$$u_{n+1} = A(\theta_n)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{3}$$

with $(u_n,\theta_n)\in\mathbb{R}^2\times\mathbb{T}$ for all $n\in\mathbb{Z}$ is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|Im\ \theta|<\delta$. Moreover the Floquet matrix can be chosen to be of the form

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{4}$$

for some $c\in\mathbb{R}$.

Proof: Triangularization of Cocycles

The transformation

$$Z(\theta) = \begin{pmatrix} v_1(\theta) & -v_2(\theta)/d(\theta) \\ v_2(\theta) & v_1(\theta)/d(\theta) \end{pmatrix}, \qquad d = v_1^2 + v_2^2$$

conjugates (A, ω) with (B^1, ω), with

$$B^1(\theta) = \begin{pmatrix} 1 & b_{12}^1(\theta) \\ 0 & 1 \end{pmatrix},$$

for some analytic $b_{12}^1 : \mathbb{T} \to \mathbb{R}$. Let $y_{12} : \mathbb{T} \to \mathbb{R}$ be a real analytic solution of

$$y_{12}(2\pi\omega + \theta) - y_{12}(\theta) = b_{12}^1(\theta) - [b_{12}^1], \qquad \theta \in \mathbb{T},$$

where $[b_{12}^1]$ is the average of b_{12}^1 (here we use the Diophantine and analyticity conditions). Then the transformation

$$Y(\theta) = \begin{pmatrix} 1 & y_{12} \\ 0 & 1 \end{pmatrix}$$

conjugates (B^1, ω) with (B, ω), where $B = [B^1]$.

Absence of coexisting quasi-periodic solutions

If $B = I$ ($c = 0$) there are two linearly independent real analytic quasi-periodic solutions with frequency ω of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos(2\pi\omega n + \phi)x_n = \frac{2a}{b}x_n, \qquad n \in \mathbb{Z}.$$

Passing to the dual, this means that

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n)x_n = ax_n, \qquad n \in \mathbb{Z}.$$

has two linearly independent solutions in $l^2(\mathbb{Z}) \Rightarrow$ Contradiction!.

If $B \neq I$ ($c \neq 0$), we are going to show that $2a/b$ is the endpoint of a non-collapsed gap of $\sigma_{4/b}$.

Lemma 4 *Assume ω Diophantine and a as above. If $|\alpha| \neq 0$ is small enough and $c\alpha < 0$ then*

$$\frac{2a}{b} + \alpha \notin \sigma_{4/b}$$

so that $2a/b$ is the endpoint of a non-collapsed spectral gap.

How do we prove that a point is in the resolvent set?

Yet Another Characterization of the Spectrum

Theorem 5 (Johnson, 1981) *Let ω be irrational. Then $a_0 \in \rho_{b_0} = \mathbb{R} - \sigma_{b_0}$ if, and only if, the skew product*

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 - b_0 \cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

has an exponential dichotomy: it is reducible to constant coefficients with a hyperbolic Floquet matrix.

Then we restate our previous lemma as

Lemma 6 *If $|\alpha| \neq 0$ is small enough and $c\alpha < 0$ then*

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} + \alpha - \frac{4}{b} \cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

has an exponential dichotomy, and, therefore $2a/b$ is the endpoint of a non-collapsed spectral gap of $\sigma_{4/b}$

Remark 7 *Sometimes exponential dichotomy is called also uniform hyporbolicity in this context.*

Proof: exponential dichotomy in practice

We know that

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

is reducible to constant coefficients with Floquet matrix B by means of a transformation $Z : \mathbb{T} \to SL(2, \mathbb{R})$. This conjugates

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} + \alpha - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

with

$$\begin{aligned} y_{n+1} &= \begin{pmatrix} 1 + \alpha\left(z_{11}z_{12} - c z_{11}^2\right) & c + \alpha\left(-c z_{11}z_{12} + z_{12}^2\right) \\ -\alpha z_{11}^2 & 1 - \alpha z_{11}z_{12} \end{pmatrix} y_n, \\ \theta_{n+1} &= \theta_n + 2\pi\omega, \end{aligned} \tag{5}$$

being $y_n \in \mathbb{R}^2$ some new variables and z_{ij} the elements of Z.

Remark 8 *The conjugation to constant coefficients when $\alpha = 0$ implies that $z_{11} = \mu\tilde{\psi}$, with $\mu \neq 0$.*

Averaging of quasi-periodic skew-products

Restricting the domains in Im θ and α, we can conjugate system (5) to

$$\begin{aligned} y_{n+1} &= \left(\begin{pmatrix} 1+\alpha\left([z_{11}z_{12}]-c[z_{11}^2]\right) & c+\alpha\left(-c[z_{11}z_{12}]+[z_{12}^2]\right) \\ -\alpha[z_{11}^2] & 1-\alpha[z_{11}z_{12}] \end{pmatrix} + M \right) y_n \\ \theta_{n+1} &= \theta_n + 2\pi\omega \end{aligned} \quad (6)$$

where,

- $[\cdot]$ is the average.
- $M : \mathbb{T} \to gl(2, \mathbb{R})$ is real analytic in θ and α.
- M is of order α^2 at $\alpha = 0$.

If we forget about α^2-terms,

$$\begin{aligned} y_{n+1} &= \left(\begin{pmatrix} 1+\alpha\left([z_{11}z_{12}]-c[z_{11}^2]\right) & c+\alpha\left(-c[z_{11}z_{12}]+[z_{12}^2]\right) \\ -\alpha[z_{11}^2] & 1-\alpha[z_{11}z_{12}] \end{pmatrix} \right) y_n \\ \theta_{n+1} &= \theta_n + 2\pi\omega \end{aligned} \quad (7)$$

is a system with constant coefficients whose trace is $2 - c\alpha[z_{11}^2]$ so that, if $c\alpha < 0$, trace > 2, there is exponential dichotomy.

but... what about the whole system?

A sufficient criterion for exponential dichotomy

One has the following analog of Gerschgorin formula for quasi-periodic cocycles:

Theorem 9 (Coppel, 1978) *Let* ω *be irrational,* $\Lambda = diag\ (\lambda, -\lambda)$ *a diagonal matrix. Let* $N : \mathbb{T} \to sl(n, \mathbb{R})$ *with*

$$\max_{1 \le i,j \le 2} \max_{\theta \in \mathbb{T}} |N_{i,j}| < \frac{|\lambda|}{3}.$$

Then the cocycle $(\exp(\Lambda + N), \omega)$ *has an exponential dichotomy.*

To apply it to our case, note that, for α small enough

$$\begin{pmatrix} 1 + \alpha\left([z_{11}z_{12}] - c[z_{11}^2]\right) & c + \alpha\left(-c[z_{11}z_{12}] + [z_{12}^2]\right) \\ -\alpha[z_{11}^2] & 1 - \alpha[z_{11}z_{12}] \end{pmatrix} + M = e^{A(\theta,\alpha)},$$

where, by means of a computation, it is seen that

$$A(\theta,\alpha) = \begin{pmatrix} \alpha\left([z_{11}z_{12}] - \frac{c}{2}[z_{11}^2]\right) & c + \alpha\left(-c[z_{11}z_{12}] + [z_{12}^2]\right) \\ -\alpha[z_{11}^2] & -\alpha\left([z_{11}z_{12}] - \frac{c}{2}[z_{11}^2]\right) \end{pmatrix} + \tilde{M}(\theta,\alpha),$$

where $\tilde{M}$ is of order α^2 at $\alpha = 0$. After a change of variables for $c\alpha < 0$, the result follows applying Coppel theorem.

Reducibility close to constant coefficients

Theorem 10 (Eliasson (1992)) *Assume that ω is Diophantine with constants c and r. Then there is a constant $C(c,r)$ such that, if $|b| < C(c,r)$ and rot(a,b) is either rational,*

$$\text{rot}(a,b) = \{k\omega\}/2, \text{ for some } k \in \mathbb{Z},$$

or Diophantine,

$$\left|\text{rot}(a,b) - \frac{\{k\omega\}}{2}\right| = \frac{1}{2}\left|k(a,b) - \{k\omega\}\right| \geq \frac{K}{|k|^\tau},$$

then

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - b\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \qquad (8)$$

on $\mathbb{R}^2 \times \mathbb{T}$ is reducible to constant coefficients, with Floquet matrix B. Moreover, if a is at an endpoint of a spectral gap of σ_b, then

$$\text{trace } B = \pm 2$$

and

$$B = \pm I \Leftrightarrow \text{the gap collapses.}$$