Reducibility of Quasi-Periodic Skew-Products and the Spectrum of Schrödinger Operators

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Hill's equation with quasi-periodic forcing

Our starting point is the second-order differential equation

$$x'' + (a - bq(t))\, x = 0, \tag{1}$$

- a and b are real parameters.
- The *forcing* q is quasi-periodic with *frequency* $\omega \in \mathbb{R}^d$, $q(t) = Q(\omega t)$, where $Q : \mathbb{T}^d \to \mathbb{R}$ is continuous (usually real analytic) and ω is *rationally independent*,

$$\langle \mathbf{k}, \omega \rangle = k_1\omega_1 + \ldots + k_d\omega_d \neq 0 \qquad \text{for all } \mathbf{k} \notin \mathbb{Z}^d \setminus \{\mathbf{0}\}.$$

- It is a generalization of the classical Hill's equation, where the forcing is periodic, introduced by George Hill in the 19th century.
- It is a prototypical example of the linearization of a non-linear system with quasi-periodic forcing around a fixed point, or a Hamiltonian system around an invariant torus.

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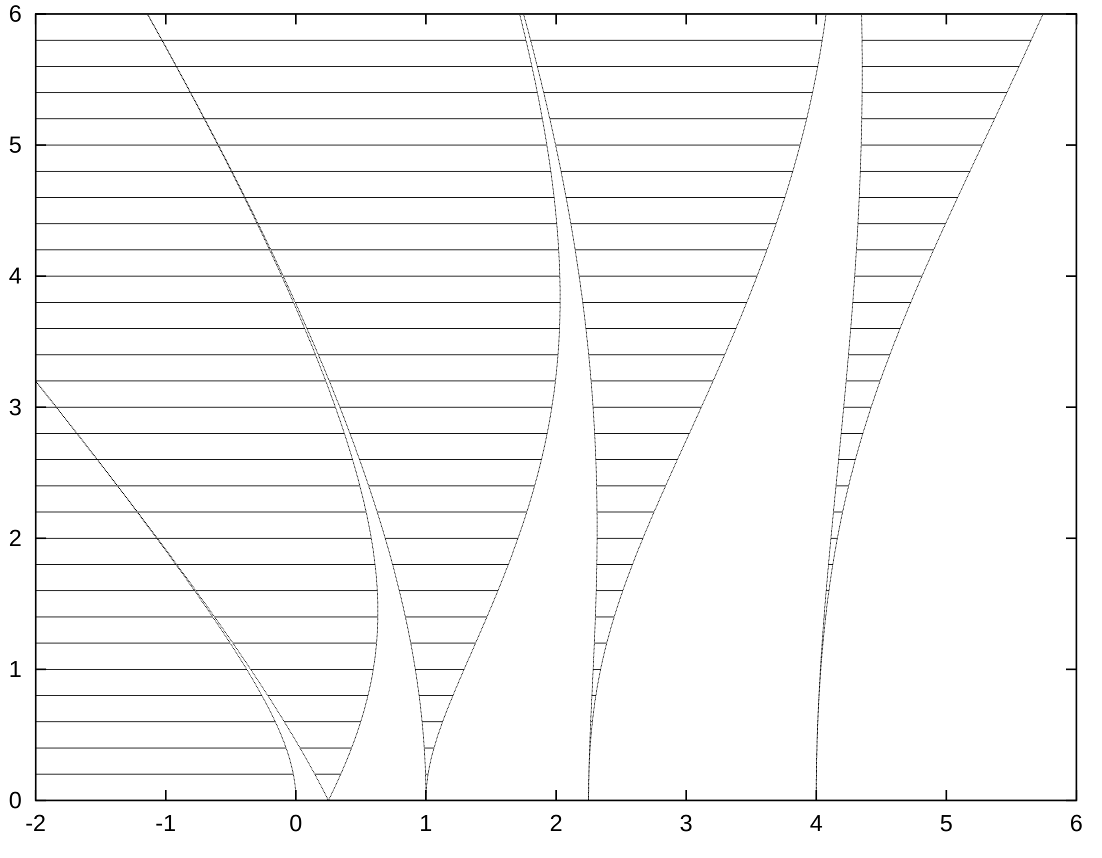
Example: Resonance Tongues of Mathieu's equation

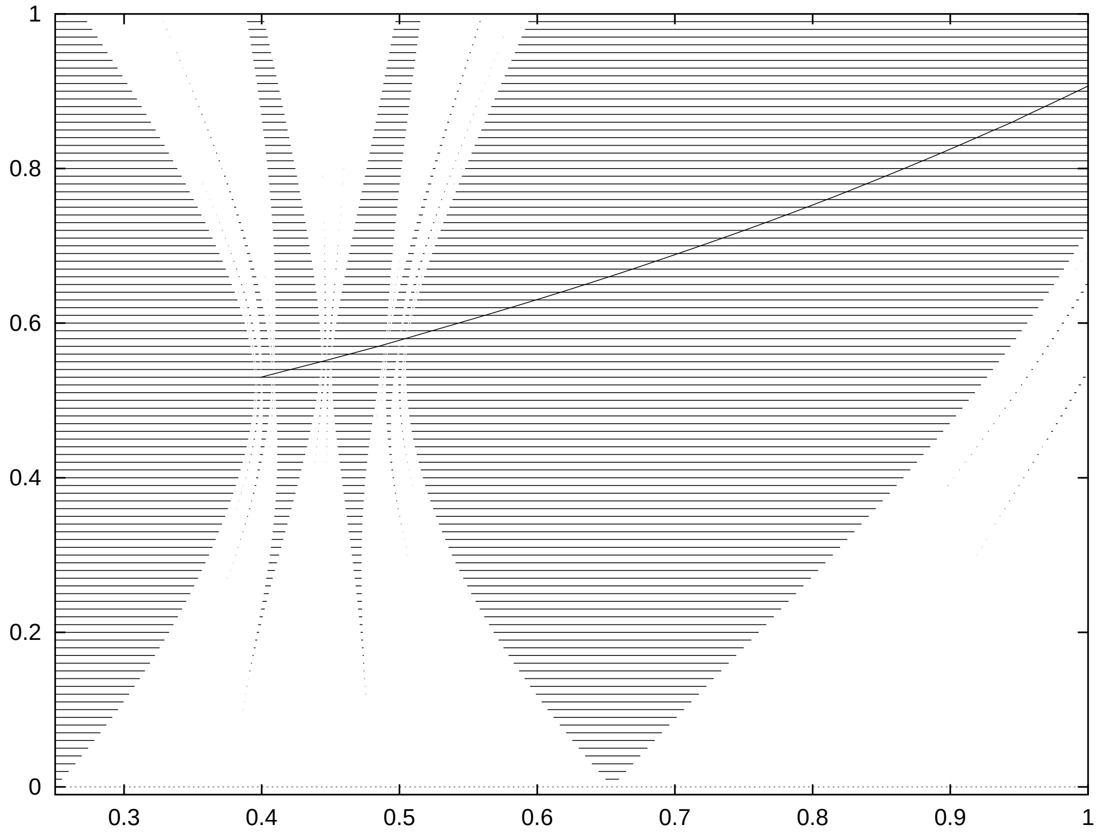
A classical example of Hill's equation (compare with Ince, 1927) is Mathieu's equation,

$$x'' + (a + b\cos(t))\,x = 0$$



The quasi-periodic case

If q is quasi-periodic, for instance $q(t) = \cos(t) + \cos\left(\frac{1+\sqrt{5}}{2}t\right)$, (see Broer & Simó, 1998)



Problem: Understand the structure of stable and unstable zones in the (a, b)-plane in the quasi-periodic case.

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Quasi-Periodic Schrödinger Operators

Hill's equation can be written as an eigenvalue equation

$$H^c_{bQ,\omega,\phi} x = ax$$

of the *quasi-periodic Schrödinger operator*:

$$\left(H^c_{bQ,\omega,\phi} x\right)(t) = -x''(t) + bQ(\omega t + \phi)x(t).$$

- $H^c_{bQ,\omega,\phi}$ is *essentially self-adjoint* on $L^2(\mathbb{R})$ and *unbounded*.
- Parameter a is a *spectral parameter*, called the *energy*, and q is called the *potential*.
- These operators are used in Quantum Physics, notably in the comprehension of the *Quantum Hall Effect* and the electronic properties of *quasi-crystals*.

Problem: Which is the dependence of the spectrum of these operators as a function of b?

Problem: Can we say something about the spectrum of these operators studying the dynamics of their eigenvalue equations ?

A dynamical approach to Hill's equation

To study the dynamics of Hill's equation it is convenient to write it down as a an *autonomous*, *first-order sytem*:

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ bQ(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega,$$

where $\theta \in \mathbb{T}^d$. Such a dynamical system on $\mathbb{R}^2 \times \mathbb{T}^d$ is called a *linear skew-product flow*. In general, this is of the form

$$z' = A(\theta)z, \qquad \theta' = \omega$$

where $z \in \mathbb{R}^n$ and $A : \mathbb{T}^d \to gl(n, \mathbb{R})$ is an $n \times n$ matrix. The simplest example is given by skew-products with *constant coefficients*

$$z' = Az, \qquad \theta' = \omega$$

where A is a constant matrix. Such systems can be directly integrated,

$$z(t) = e^{tA}z(0), \qquad \theta(t) = \omega t + \theta(0).$$

Reducibility of Quasi-Periodic Skew-Products Flows

A quasi-periodic skew-product

$$x' = A(\theta)x, \qquad \theta' = \omega, \tag{2}$$

is *reducible to constant coefficients* if there is a map $Z : \mathbb{T}^d \to GL(n, \mathbb{R})$ such that the change of variables

$$(z, \theta) = (Z(\theta)y, \theta)$$

transforms (2) into

$$y' = By, \qquad \theta' = \omega. \tag{3}$$

where B does not depend on θ. The matrix B is called a *Floquet* matrix and Z a *reducing transformation*. Sometimes it may be necessary to “halve the frequency”.

Problem: Which skew-products like (2) are *reducible* to constant coefficents by means of a quasi-periodic transformation?

Problem: In the Hill case, which implications has the reducibility of the skew-product for the spectrum of the corresponding Schrödinger operator?

The Discrete Case: Harper-like Equations

A discretization of Hill's equation (after some rearrangements) leads to the difference equation

$$x_{n+1} + x_{n-1} + bv(n)x_n = ax_n,$$

where

- $(x_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers.
- a, b are real parameters
- $(v(n))_{n\in\mathbb{Z}}$ is a *quasi-periodic sequence*. That is, there is a continuous $V : \mathbb{T}^d \to \mathbb{R}$ (usually assumed real analytic) such that

$$v(n) = V\left(2\pi\omega n\right), \qquad n \in \mathbb{Z}$$

where $\omega \in \mathbb{R}^d$ is *nonresonant*:

$$\langle \mathbf{k}, \omega \rangle \notin \mathbb{Z} \qquad \text{for all } \mathbf{k} \notin \mathbb{Z}^d \setminus \{\mathbf{0}\}.$$

Dynamical and Spectral Approaches

Like Hill's equation Harper-like equations can be lifted to first-order systems on $\mathbb{R}^2 \times \mathbb{T}^d$,

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - bV(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{4}$$

which are called *discrete quasi-periodic skew-products*.

They can also be seen as eigenvalue equations

$$H^d_{bV,\omega,\phi} x = x$$

of the following *discrete quasi-periodic Schrödinger operators*,

$$(H^d_{bV,\omega,\phi} x)_n = x_{n+1} + x_{n-1} + bV(2\pi\omega n + \phi) x_n$$

which is a bounded and self-adjoint operator of $l^2(\mathbb{Z})$.

The Ten Martini Problem

The *Almost Mathieu* operator

$$(H_{b,\omega,\phi}^{AM} x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)\, x_n$$

was introduced in physical studies (Harper 1955, Azbel 1964). Its eigenvalue equation is the *Harper equation*

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)\, x_n = a x_n$$

Problem: Kac-Simon (1982) posed the *Ten Martini Problem*: prove that the spectrum of the Almost Mathieu operator is a *Cantor set* for nonresonant ω and $b \neq 0$.

Note that this is a *nonperturbative* problem: the values of b for which it holds do not depend on the specific properties of ω (rather than being irrational).

Problem: Can we produce *nonperturbative* results for other models as well?

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Quasi-Periodic Schrödinger operators

Consider a quasi-periodic Schrödinger operator, either continuous

$$\left(H^c_{Q,\omega,\phi}x\right)(t) = -x''(t) + Q(\omega t + \phi)x(t) \qquad \text{on } L^2(\mathbb{R})$$

or discrete

$$(H^d_{V,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \phi) \qquad \text{on } l^2(\mathbb{Z}).$$

If ω is rationally independent (resp. nonresonant) and Q (resp. V) is continuous then the spectrum of $H^c_{Q,\omega,\phi}$ (resp. $H^d_{V,\omega,\phi}$) does not depend on $\phi \in \mathbb{T}^d$. Therefore we write

$$\sigma^c(Q,\omega) = \mathrm{Spec}\left(H^c_{Q,\omega,\phi}\right)$$

and

$$\sigma^d(V,\omega) = \mathrm{Spec}\left(H^d_{V,\omega,\phi}\right).$$

In both cases the spectrum can be characterized in terms of dynamical properties of the corresponding skew-products on $\mathbb{R}^2 \times \mathbb{T}^d$. Let us focus on the continuous case.

The rotation number (continuous case)

Let Q be continuous, ω rationally independent and x a nontrivial solution of

$$x'' + (a - bQ(\omega t + \phi))\, x = 0. \tag{5}$$

Johnson & Moser (1982) proved that the limit exists and it is independent of ϕ and x,

$$\mathrm{rot}^c(a - bQ, \omega) := \lim_{t \to \infty} \frac{\arg\left(x'(t) + \mathrm{i}x(t)\right)}{t}$$

and it is called the *rotation number* of (5). Consider the map

$$a \in \mathbb{R} \mapsto \mathrm{rot}^c(a - bQ, \omega) \in [0, +\infty).$$

- It is nondecreasing. Moreover $a \in \sigma^c(bQ, \omega) \Leftrightarrow$ it is a point of increase.
- (Gap Labelling) In the intervals of constancy, there is a $\mathbf{k} \in \mathbb{Z}^d$ such that

$$\mathrm{rot}^c(a - bQ) = \frac{\langle \mathbf{k}, \omega \rangle}{2}.$$

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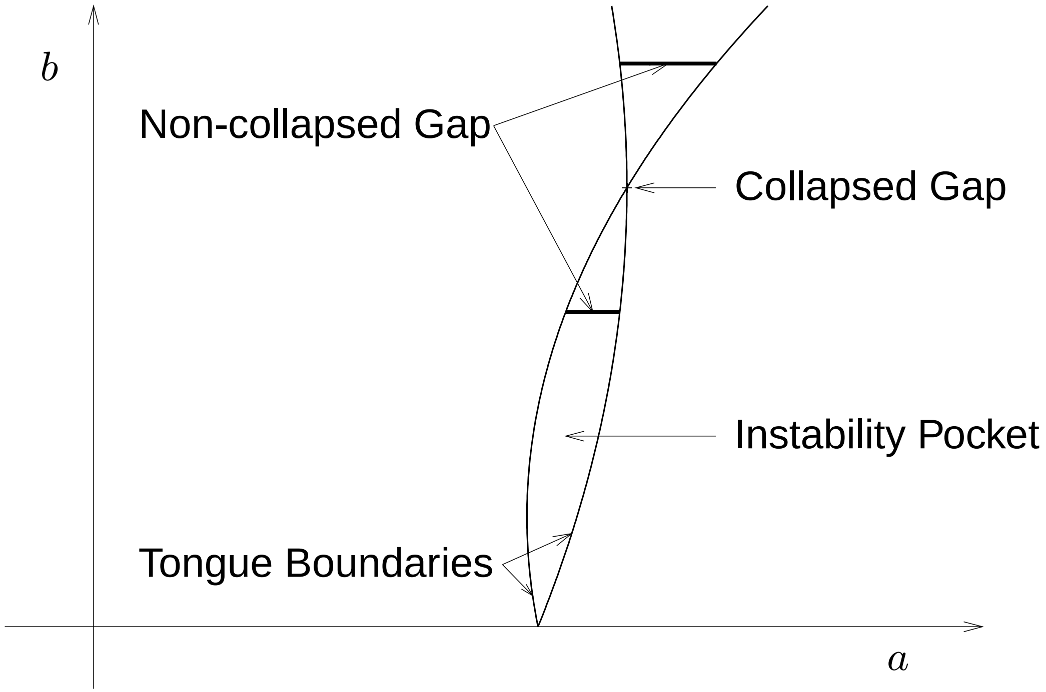
Resonance tongues

Let $\mathbf{k} \in \mathbb{Z}^d$ with $\alpha_0 = \langle \mathbf{k}, \omega \rangle / 2 \geq 0$. The *resonance tongue* associated to $\mathbf{k}$ is the set of those $(a, b) \in \mathbb{R}^2$ such that

$$\mathrm{rot}^c(a - bQ, \omega) = \alpha_0$$



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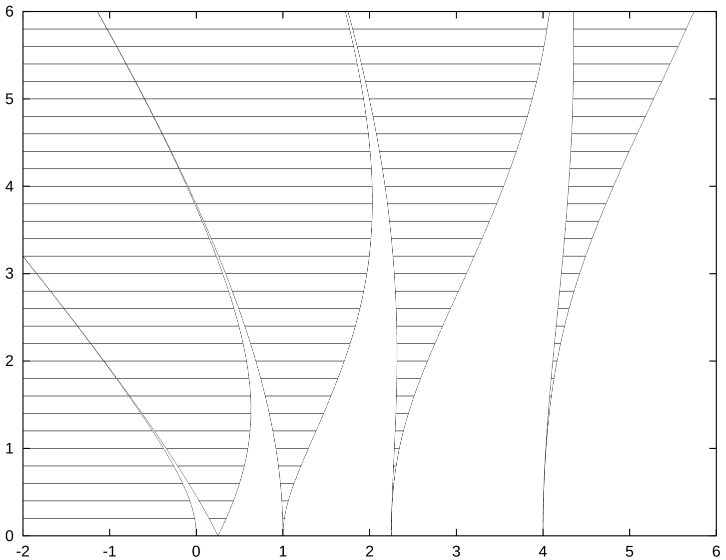
Cantor Spectrum and Denseness of Tongues

Let $\omega \in \mathbb{R}^d$ be rationally independent. Define the set of *rational rotation numbers* (with respect to ω) or *ω-rationals* as

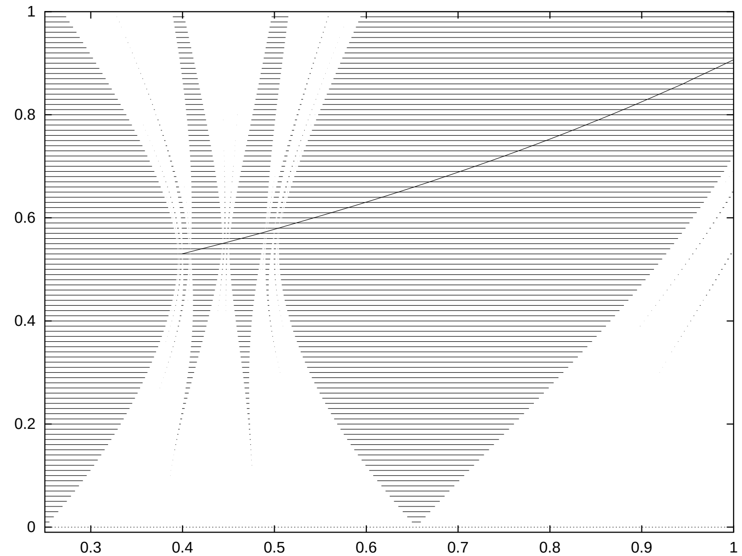
$$\mathcal{M}_+(\omega) = \left\{ \langle \mathbf{k}, \omega \rangle / 2; \mathbf{k} \in \mathbb{Z}^d \text{ and } \langle \mathbf{k}, \omega \rangle \geq 0 \right\} \subset [0, \infty).$$



Periodic Case ($d = 1$)

$\mathcal{M}_+(\omega)$ has no accumulation points.

Tongues are separated one from each other



Quasi-Periodic Case ($d \geq 2$)

$\mathcal{M}_+(\omega)$ is dense in $[0, +\infty)$

Tongues are dense in the (a, b)-plane

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Reducibility in Hill's equation

Eliasson (1992) proved that if in

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ bQ(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega, \tag{6}$$

- ω is *strongly rationally independent*, i.e. there are positive constants c and τ such that

$$|\langle \mathbf{k}, \omega \rangle| \geq c|\mathbf{k}|^{-\tau} = c\,(|k_1| + \ldots + |k_d|)^{-\tau}, \qquad \text{for all } \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$$

- Q is real analytic, with

$$|Q|_\rho := \sup_{|\mathrm{Im}\,\theta| < \rho} |Q(\theta)|\,.$$

Then there is a constant $C = C(c, \tau, \rho, |Q|_\rho)$ such that, if $|b| < C$,

- System (6) is reducible for Lebesgue-almost every value of a .
- If (a, b) is at a tongue boundary, (6) is reducible. The Floquet matrix B satisfies $B^2 = 0$. Moreover, $B = 0 \Leftrightarrow$, a is a collapsed gap of $\sigma^c(a - bQ, \omega)$.

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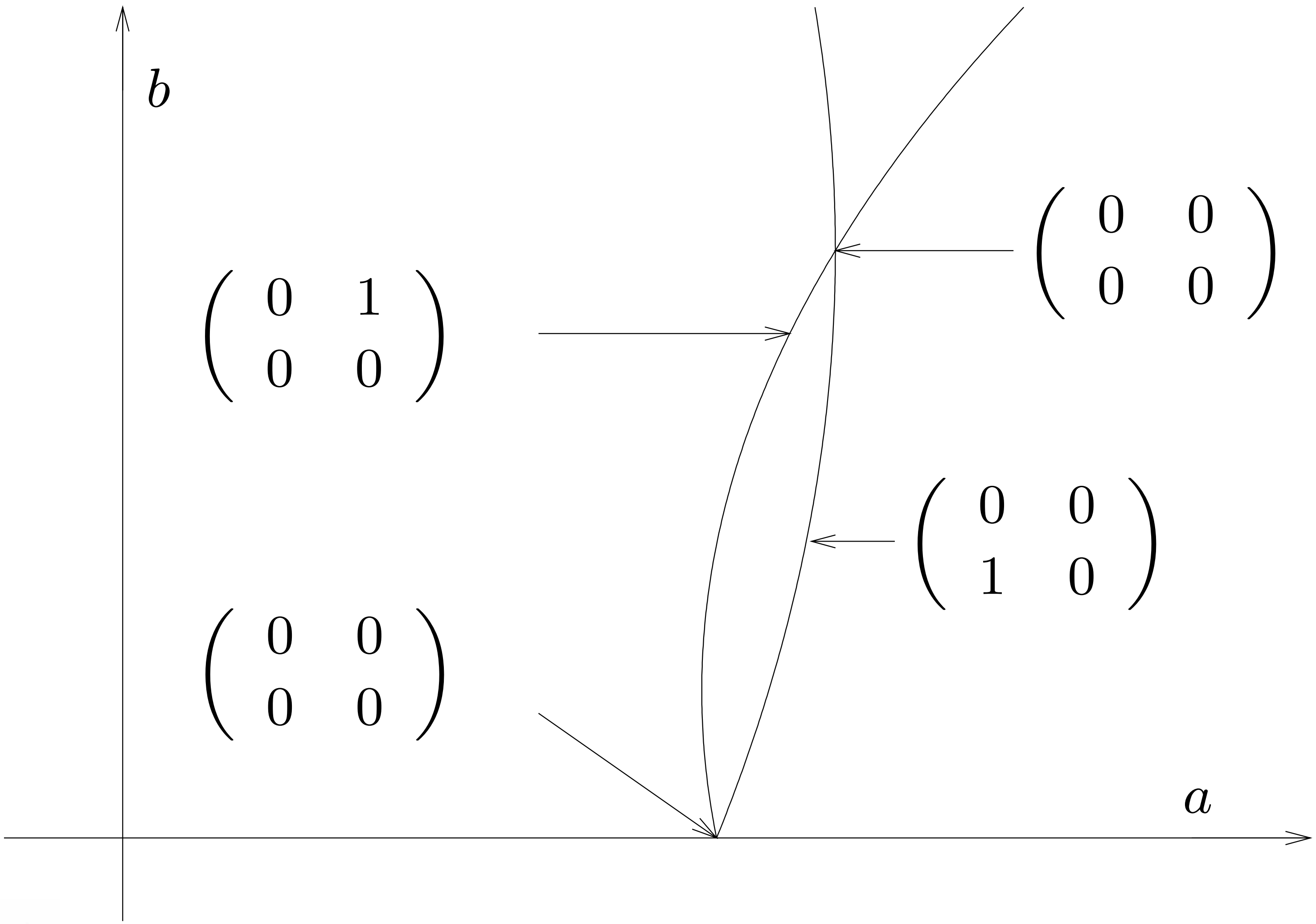
Reducibility at tongue boundaries

Examples of Floquet matrices at tongue boundaries:



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The local situation at tongue boundaries

Assume ω Diophantine, Q real analytic and $|b_0| < C$. Then, by Eliasson's result,

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ bQ(\theta) - a & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega, \tag{7}$$

is reducible to constant coefficients if $(a, b) = (a_0, b_0)$ and a_0 is at a tongue boundary by a quasi-periodic change of variables

$$\begin{pmatrix} x' \\ x \end{pmatrix} = Z\left(\frac{\omega t}{2}\right) y \tag{8}$$

with $Z : \mathbb{T}^d \to SL(2, \mathbb{R})$ real analytic such that the Floquet matrix $B^2 = 0$. The transformation (8) tranforms (7) into

$$y' = (B + P(\theta, \mu))\, y, \qquad \theta' = \omega.$$

where $P \in sl(2, \mathbb{R})$ is real analytic in θ and the new *local parameters*

$$\mu = (a - a_0, b - b_0).$$

Normal Forms around Tongue Boundaries

After r steps of Birkhoff Normal Form, we conjugate the system to

$$y' = \left(B + \sum_{k=1}^{r} B_k(\mu) + P^{r+1}(\theta,\mu) \right) y, \qquad \theta' = \omega. \tag{9}$$

1. $B_k(\mu) \in sl(2,\mathbb{R})$ is homogeneus and of order k in μ for $k = 1, \ldots, r$.
2. $P^{r+1} \in sl(2,\mathbb{R})$ is real analytic for $|\mu|$ and $|\mathrm{Im}\,\theta|$ small enough, with $|P^{r+1}| = O_{r+1}(\mu)$.

Even if (9) is not in constant coefficients we prove that *the equation*

$$\det \left(B + \sum_{k=1}^{r} B_k(\mu) \right) = 0$$

determines the Taylor expansion up to order r of tongue boundaries around (a_0, b_0), $a_{1,2} = a_{1,2}(b)$ and that these are C^∞-functions (for $|b|$ small, Q analytic and ω s.r.i). This is applied to the existence of pockets in several families of Hill's equations.

The problem of analyticity of tongue boundaries

- The analyticity of tongue boundaries would have interesting consequences:

  Analyticity of boundaries
  Opening of tongues at $b = 0$ $\Rightarrow$ Genericity of opening of gaps.

- In the periodic case the analyticity is a consequence of the separation between tongues, which makes the Normal Form reduction locally convergent.
- In the quasi-periodic case the genericity of Cantor spectrum makes the Normal Form generically divergent. In fact, this can be used to show that the Birkhoff Normal form of a quasi-periodic Hamiltonian with fixed frequencies and linear part is generically divergent.
- The proof of analyticity of tongue boundaries in the quasi-periodic case needs different techniques: *for which values of* μ *is*

$$y' = (B + P(\theta, \mu))\, y, \qquad \theta' = \omega.$$

reducible to

$$y' = By, \qquad \theta' = \omega \qquad ?$$

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Counterterms in Hill's equation

Following Moser (1967), we will find a time-free matrix, $M = M(\mu) \in sl(2, \mathbb{R})$, called *counterterm*, depending analytically on μ such that

$$z' = (B + P(\theta, \mu) - M(\mu))\, z, \qquad \theta' = \omega,$$

is reducible with Floquet matrix B. The shape of M depends on B, for instance

$$M(\mu) = \begin{pmatrix} m_{11}(\mu) & m_{12}(\mu) \\ m_{21}(\mu) & -m_{11}(\mu) \end{pmatrix} \quad \text{if} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$M(\mu) = \begin{pmatrix} 0 & 0 \\ m_{21}(\mu) & 0 \end{pmatrix} \quad \text{if} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the latter case (where a_0 is an endpoint of a noncollapsed spectral gap) *the equation* $m_{21}(a - a_0, b - b_0) = 0$ *determines the boundary of the resonance tongue around* (a_0, b_0).

Analyticity of tongue boundaries

- Under the assumption of ω strongly rational independent and Q real analytic, we show that *tongue boundaries are real analytic functions* $a_{1,2} = a_{1,2}(b)$ *in a neighbourhood of* $b = 0$.
- In the case of collapsed gap, this requires a more refined version of the previous methodology, allowing the presence of *scaling factors*.
- As a consequence, we prove that *Schrödinger operators with small, real analytic, quasi-periodic potentials have generically all gaps open*. In fact, we prove that, *as long as tongues open when* $b = 0$ *at any finite order, gap opening happens for all small values of* b *except for a countable set*.
- These theorems are proved in a unified context of *Lie algebras*. We study the existence of a counterterm $M = M(\mu)$ such that

$$y' = \chi(\mu)^k \left(B + \chi(\mu)P(\theta, \mu) - \chi(\mu)M(\mu)\right) y, \qquad \theta' = \omega$$

  is reducible with Floquet matrix $\chi(\mu)^k B$, where B, P and M lie in some matrix Lie algebra of $gl(n, \mathbb{R})$. B and ω need to satisfy some conditions.
- We also study the cases of $so(3, \mathbb{R})$ and $sp(n, \mathbb{R})$.

The Ten Martini Problem Revisited

The best studied quasi-periodic Schrödinger operator is the *Almost Mathieu*:

$$(H^{AM}_{b,\omega,\phi}x)_n = x_{n+1} + x_{n-1} + \cos(2\pi\omega n + \phi)\, x_n$$

for $b \in \mathbb{R}$ and ω nonresonant. The eigenvalue equation is called *Harper's equation*

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)\, x_n = a x_n, \qquad n \in \mathbb{Z}.$$

Simon-Kac (1981): Prove that $\sigma^{AM}(b,\omega) = \operatorname{Spec}\left(H^{AM}_{b,\omega,\phi}\right)$ is a Cantor set if $b \neq 0$ and ω nonresonant.

Corollary: *If ω is strongly nonresonant,*

$$|\sin(\pi k\omega)| \geq \frac{c}{|k|^{\tau}}, \qquad k \in \mathbb{Z} \setminus \{0\}$$

for some constants $b \neq 0, \pm 2$, then $\sigma^{d}_{AM}(b,\omega)$ is a Cantor set.

Nonperturbative Localization and Aubry Duality

Theorem (Jitomirkskaya, 1999): *If ω is strongly nonresonant and $|b| > 2$ then the operator $H_{b,\omega,0}$ has only* pure-point spectrum *with* exponentially decaying eigenfunctions.

Let $|b| > 2$. Then there is a dense subset of $\sigma AM(b,\omega)$, $\sigma_{pp}^{AM}(b,\omega,\phi)$, such that for a in this set Harper's equation has a solution $\psi = (\psi_k)_{k\in\mathbb{Z}}$ which decays exponentially in k. Its Fourier transform is a real analytic function of $\mathbb{T}$,

$$\tilde{\psi}(\theta) = \sum_{k\in\mathbb{Z}} \psi_k e^{\mathrm{i}k\theta},$$

and the *quasi-periodic Bloch wave* $x_n = \tilde{\psi}\,(2\pi\omega n)$ is a solution of the Harper equation

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos\left(2\pi\omega n\right) x_n = \frac{2a}{b} x_n.$$

The invariance under Fourier Transform is known as *Aubry duality*,

$$(a,b) \mapsto \left(\frac{2a}{b}, \frac{4}{b}\right).$$

Ince's argument for the Lack of Coexisting Bloch Waves

How to end the proof?

- The existence of such quasi-periodic Bloch waves (with frequency ω) implies that $2a/b$ is at the endpoint of a spectral gap of $H_{4/b,\omega,\phi}$.
- This gap is collapsed $\Leftrightarrow$ the Harper equation for $(2a/b, 4/b)$ has another linearly independent Bloch wave.
- If the gap was collapsed, by duality there would be two linearly independent and exponentially localized solutions of the original Harper equation.
- This is a contradiction with the limit-point character of the cosine: two $l^2(\mathbb{Z})$ solutions of a Harper equation cannot coexist.
- Since the values of a's for which this holds are dense in the spectrum and they are endpoints of noncollapsed gaps $\Rightarrow$ *the spectrum of the Almost Mathieu operator is a Cantor set!*

Non-Perturbative Reducibility

Jitomirskaya's result was extended by Jitomirskaya & Bourgain (2002) to deal with general real analytic potentials. This can be used to produce a nonperturbative version of Eliasson's Theorem for discrete quasi-periodic Schrödinger skew-products:

Theorem: *Fix $\rho > 0$. There is a constant $\varepsilon_0 = \varepsilon_0(\rho)$ such that, if $V : \mathbb{T} \to \mathbb{R}$ is real analytic,*

$$|V|_\rho < \varepsilon_0,$$

and ω is strongly nonresonant then the quasi-periodic skew-product

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - V(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega.$$

is reducible to constant coefficients (by means of a real analytic and quasi-periodic tranformation) for Lebesgue-almost every $a \in \mathbb{R}$.

Perturbative vs. Nonperturbative results

- This is a nonperturbative reducibility result: the constant ε does not depend on the precise Diophantine conditions.

- It is specific of potentials V on $\mathbb{T}$. It is not true for $\mathbb{T}^d$ with $d \geq 2$.

- KAM-like methods usually produce *perturbative* results, where the bounds depend on the Diophantine constant. On the contrary there is no limitation on d.

- Compared to Eliasson's theorem, the nonperturbative version above has the limitation that the set of reducible values of a is not determined in terms of the rotation number.

Conclusions

- The structure of resonance tongues in quasi-periodic Hill's equation has been understood for $|b|$ small, analytic Q and ω Diophantine. Differences and analogies with the periodic case have been derived.
- This has been applied to study the spectrum of Schrödinger operators with quasi-periodic potential. Gap opening and closing has been explained in the perturbative situation.
- With the analyticity of tongue boundaries, the genericity of gap opening, and Cantor spectrum in particular, has been established. This is applied to the generic divergence of Birkhoff Normal Form for quasi-periodic Hamiltonians.
- The Ten Martini Problem has been proved. We have also proved that, for $|b|$ small or large enough all spectral gaps are open, a partial answer to the Strong Ten Martini Problem.
- We have shown the connection between reducibility and localization results by means of duality. This has allowed us to produce a nonperturbative version of Eliasson's Theorem for discrete quasi-periodic Schrödinger operators with one frequency.

Outlook

- Comprehension of the structure of the spectrum of quasi-periodic Schrödinger operators through an analysis of the corresponding eigenvalue equations in higher dimensions and more frequencies.

- The problem of the regularity of the boundaries of hiperbolicity (or partial hyperbolicity domains) is interesting in the higher dimensional context. For this, more tools will be needed.

- Applications of the linear quasi-periodic theory to the nonlinear case. In many cases these operators are linearizations of nonlinear operators. The different spectral behaviours in the linear case may have implications for the nonlinear case.

- And more unexpected applications...

Normal Forms and Structure of resonance tongues

The previous procedure is effective for $b = 0$.

Proposition: *If Q is real analytic and ω strongly rationally independent, the boundaries of the $\mathbf{k}$th resonance tongue of Hill's equation are transversal when $b = 0 \Leftrightarrow$ the $\mathbf{k}$th harmonic of Q is different from zero.*

Theorem: *Consider the following quasi-periodic Hill's equation*

$$x'' + \left(a + b\left(\sum_{j=1}^{d} c_j \cos(\omega_j t) + \varepsilon \cos\left(\langle \mathbf{k}^*, \omega\rangle t\right)\right)\right) x = 0. \tag{10}$$

1. *If $\varepsilon = 0$, the order of tangency at $b = 0$ of the $\mathbf{k}^*$th resonance tongue is $\geq |\mathbf{k}^*|$, being exactly $|\mathbf{k}^*| \Leftrightarrow \omega$ does not belong to $\mathcal{A}(\mathbf{k}^*)$, a zero measure subset of the strongly rationally independent frequency vectors*
2. *If $\varepsilon \neq 0$, $\omega \notin \mathcal{A}(\mathbf{k}^*)$ and $|\varepsilon|$ is small enough, there is at least an instability pocket at the $\mathbf{k}^*$th resonance tongue with ends $b = 0$ and $b = b(\varepsilon) \neq 0$. (here ε needs a suitable sign if $|\mathbf{k}^*|$ is odd).*