The Ten Martini Problem

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The Almost Mathieu operator

The Almost Mathieu operator:

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \tag{1}$$

on $l^2(\mathbb{Z})$, where

- b is a real parameter,
- $\omega > 0$ is an irrational number and
- $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

$H_{b,\phi} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a bounded and self-adjoint operator.

The eigenvalue equation of the Almost Mathieu operator is

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z}.$$

which is a discretization of the Mathieu equation

$$x'' + (a + b\cos(t))\,x = 0.$$

studied by Ince (1922).

Alternative formulations: the dynamical formulation

- The eigenvalue equation of the Almost Mathieu operator (Harper equation):

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = a x_n, \qquad n \in \mathbb{Z}.$$

- The associated first-order system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a - b\cos(\theta_n) & -1 \\ 1 & 0 \end{pmatrix}}_{A(a-b\cos(\theta_n))} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega.$$

- The quasi-periodic skew-product on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$X_{n+1} = A(a - b\cos(\theta_n))X_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega,$$

that is,

$$X_{n+1} = \begin{pmatrix} a - b\cos(2\pi\omega n + \phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a - b\cos(\phi) & -1 \\ 1 & 0 \end{pmatrix} \cdot X_0$$

being X_0 a 2×2 matrix with $\det X_0 = 1$.

Alternative formulations: the matrix formulation of $H_{b,\phi}$

- Perturbation of a bidiagonal operator

$$\begin{pmatrix} \ddots & & & & 0 \\ & b\cos(2\pi\omega+\phi) & 1 & & \\ & 1 & b\cos(\phi) & 1 & \\ & & 1 & b\cos(-2\pi\omega+\phi) & \\ 0 & & & & \ddots \end{pmatrix}.$$

- Dilatation of a perturbation of a diagonal system (if $b \neq 0$ and $\varepsilon = 1/b$) :

$$\varepsilon\Delta + D_\phi = \begin{pmatrix} \ddots & & & & 0 \\ & \cos(2\pi\omega+\phi) & \varepsilon & & \\ & \varepsilon & \cos(\phi) & \varepsilon & \\ & & \varepsilon & \cos(-2\pi\omega+\phi) & \\ 0 & & & & \ddots \end{pmatrix}.$$

The spectrum of $H_{b,\phi}$ (Functional approach)

$$\text{Spec}(H_{b,\phi}) = \mathbb{C} - \{a \in \mathbb{C}; H_{b,\phi} - aI \ \text{ has a bounded inverse } \}$$

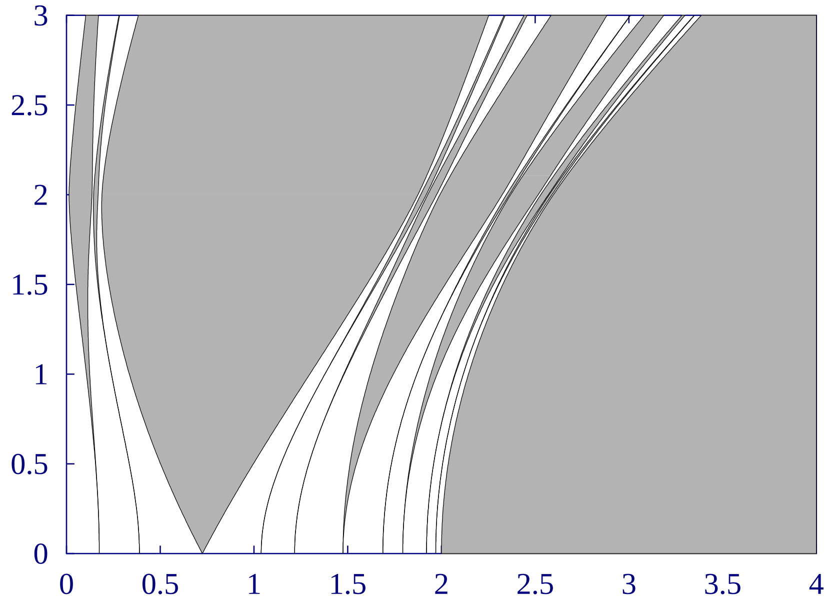
Basic facts:

- $H_{b,\phi}$ is self-adjoint $\Rightarrow$ Spec$(H_{b,\phi}) \subset \mathbb{R}$.
- $H_{b,\phi}$ is bounded $\Rightarrow$ Spec$(H_{b,\phi})$ is compact.
- $|\Delta|_{\mathcal{L}(l^2(\mathbb{Z}))} = 2$ and $|D_\phi|_{\mathcal{L}(l^2(\mathbb{Z}))} = 1 \Rightarrow$ Spec$(H_{b,\phi}) \subset [-2 - |b|, 2 + |b|]$.
- If ω is irrational $\Rightarrow$ Spec$(H_{b,\phi})$ does not depend on $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, so that we can write

$$\sigma_b = \text{Spec}(H_{b,\phi})$$

For $b = 0$, $\sigma_0 = [-2, 2] \Rightarrow$ all spectral gaps are collapsed.

However, a numerical computation shows . . .



Plot of the ten biggest spectral gaps of σ_b in the (a, b)-plane.

Question: How is σ_b as a set for $b \neq 0$?

The Ten Martini Problem

Prove that σ_b is a Cantor set for ω irrational and $b \neq 0$ (Kac-Simon, 1981)

Corollary 1 (P., Comm. Math. Phys.) *Assume that $\omega \in \mathbb{R}$ is Diophantine, that is, there exist positive constants c and $r > 1$ such that*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

for all $n \neq 0$. Then, the spectrum of the Almost Mathieu operator is a Cantor set if $b \neq 0, \pm 2$.

Why is this Cantor structure?

The spectrum of $H_{b,\phi}$ (Dynamical approach)

Assume ω irrational.

Basic property: The following properties are equivalent:

- $a_0 \in \mathbb{R}$ is in the spectrum of $H_{b,\phi}$.

- There is a $\phi \in \mathbb{T}$ such that the equation

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = a_0 x_n, \qquad n \in \mathbb{Z}. \tag{2}$$

  has a non-trival bounded solution (in particular, $\sigma_b = \mathrm{Spec}(H_{b,\phi})$ does not depend on ϕ).

- The map $a \mapsto \mathrm{rot}(a, b)$ is increasing at a_0.

The rotation number and the IDS

For a Harper's equation (2), $\text{rot}(a,b)$ is the the rotation number: the limit

$$\lim_{N\to\infty} \frac{S(N)}{2N},$$

where $S(N)$ is the number of changes of sign of a non-trivial solution of (2) in the interval $[0,N]$.

Remark 2 *The rotation number does not depend on the chosen solution nor on ϕ and it is a continuous function of (a,b) which, for b fixed is non-decreasing.*

Let

$$k_L(a,b,\phi) = \frac{1}{(L-1)} \# \left\{ \text{eigenvalues} \leq a \text{ of } H_{b,\phi}|_{\{1,\ldots,L-1\}} \right\}$$

with zero boundary conditions at both ends. Then

$$\lim_{L\to\infty} k_L(a,b,\phi) = k(a,b) = 2\text{rot}(a,b)$$

and $k(a,b)$ is the integrated density of states (IDS).

The structure of the spectrum

Question: How is the spectrum of $H_{b,\phi}$ (or how is its resolvent set?)

Gap labelling Theorem [Johnson & Moser (1982)]: If I is an open, non-void interval in the resolvent set of $H_{b,\phi}$, $\rho_b = \mathbb{R} - \sigma_b$, then there is an integer $k \in \mathbb{Z}$ such that

$$2\text{rot}(a, b) - k\omega \in \mathbb{Z}$$

for all $a \in I$. That is, $\text{rot}(a, b) = \{k\omega\}/2$ where $\{\cdot\}$ denotes the fractional part of a real number.

Definition 3 *These open intervals are called the non-collapsed spectral gaps of σ_b. If for some $k \in \mathbb{Z}$, the set*

$$\{a \in \mathbb{R}; 2rot(a, b) = \{k\omega\}\} = \{a_0\}$$

we will say that $\{a_0\}$ is a collapsed spectral gap.

If ω is irrational, the set of resonances

$$\left\{\frac{\{k\omega\}}{2};\ k \in \mathbb{Z}\right\}$$

is dense in [0,1/2].

If all spectral gaps are open, then σ_b is a Cantor set!

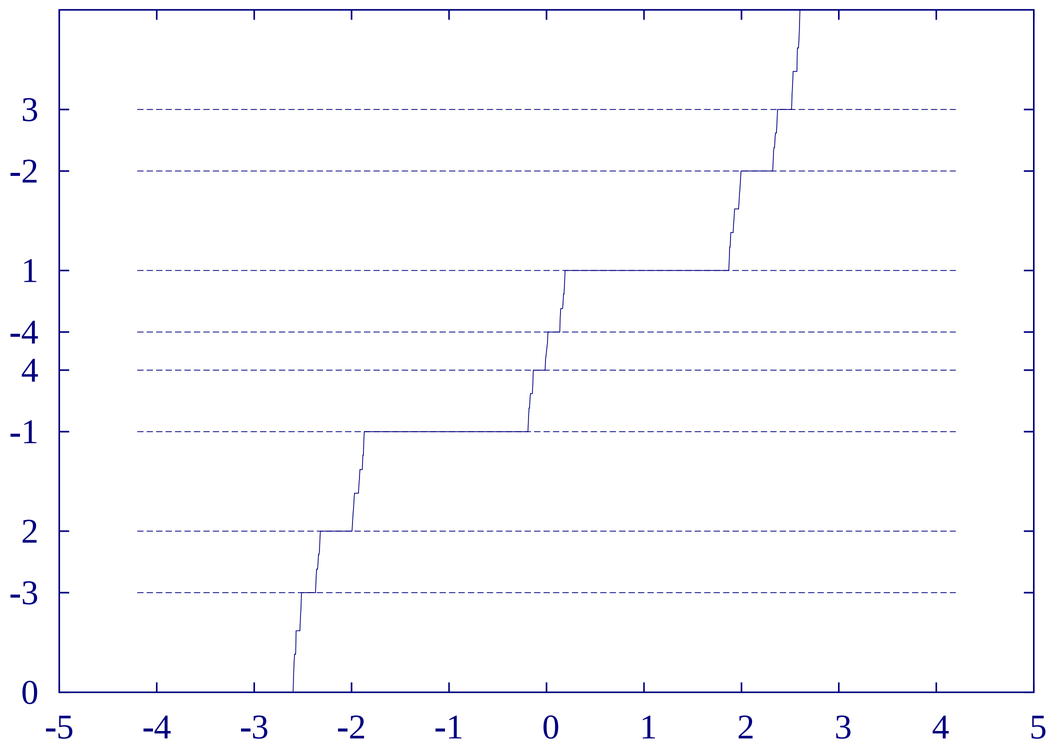
The Dry Ten Martini Problem

Prove that σ_b has all gaps open (if ω is irrational and $b \neq 0$)

Corollary 4 *Assume that $\omega \in \mathbb{R}$ is Diophantine. Then, there is a constant $C = C(\omega) > 0$ such that if $0 < |b| < C$ or $4/C < |b| < \infty$ all the spectral gaps of σ_b are open.*

Why does this hold for small and large b?

Example



Gap labelling for the operator $H_{b,\phi}$ and $b = 2$. The integer in the vertical direction corresponds to a resonance:

$$2\mathrm{rot}(a, b) = \{k\omega\}.$$

The spectral types in the perturbative situations

If $b = 0$ If $a \in [-2, 2] = \sigma_0$, the solutions of

$$x_{n+1} + x_{n-1} = a x_n, \qquad n \in \mathbb{Z}$$

are quasi-periodic with frequency $\mathrm{rot}(a, 0)$.

Expected situation for $b \approx 0$: for most values of the rotation number, the solutions of the Harper equation are quasi-periodic and analytic.

If $\varepsilon = 0$ D_ϕ has pure point spectrum for all ϕ: there is a complete orthonormal basis of eigenvectors of the operator:

$$\sigma_{pp}(D_\phi) = \left\{ \lambda^k(\phi) = \cos(2\pi\omega k + \phi); k \in \mathbb{Z} \right\}, \text{ are the eigenvalues, and}$$

$$(\psi^k)_{k\in\mathbb{Z}}, \text{ with } \psi^k_n = \delta_{k,n} \qquad \text{are the eigenvectors, and satisfy } D_\phi \psi^k = \lambda^k \psi^k.$$

Expected situation for $\varepsilon \approx 0$: for most values of ϕ, the spectrum of $\varepsilon\Delta + D_\phi$ is pure point with exponentially decaying eigenfunctions.

Duality for the Almost Mathieu operator

Idea: Take $\varepsilon > 0$ and assume that $\varepsilon\Delta + D_\phi$ has an eigenvector $\psi = (\psi_n)_{n\in\mathbb{Z}}$ with eigenvalue λ, i.e.

$$\varepsilon\,(\psi_{n+1} + \psi_{n-1}) + \cos(2\pi\omega n + \phi)\psi_n = \lambda\psi_n, \qquad n \in \mathbb{Z}.$$

and assume that it decays exponentially:

$$|\psi_n| \le A\exp(-\beta|n|), \qquad n \in \mathbb{Z}, \qquad \text{for some } A, \beta > 0.$$

Consider the function

$$\tilde{\psi}(\theta) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}, \qquad \theta \in \mathbb{T}$$

which is analytic in $|\mathrm{Im}\ \theta| < \beta$. Then, $\tilde{\psi}$ satisfies the equation

$$2\varepsilon\cos(\theta)\tilde{\psi}(\theta) + \frac{1}{2}\left(e^{-i\phi}\tilde{\psi}(\theta + 2\pi\omega) + e^{i\phi}\tilde{\psi}(\theta - 2\pi\omega)\right) = \lambda\tilde{\psi}(\theta),$$

which means that $(e^{-i\phi n}\tilde{\psi}(2\pi n\omega + \theta))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of the dual equation

$$(x_{n+1} + x_{n-1}) + b\cos(2\pi\omega n + \theta)x_n = ax_n, \qquad n \in \mathbb{Z}$$

provided

$$a = 2\lambda, \qquad b = 4\varepsilon.$$

In general we have

Theorem 5 (Avron & Simon (1983)) *For every irrational* ω*, the rotation number of*

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z}.$$

satisfies the relation

$$\mathit{rot}(a, b) = \mathit{rot}(2a/b, 4/b)$$

for all $b \neq 0$ *and* $a \in \mathbb{R}$*. In particular* $\sigma_b = \frac{b}{2}\sigma_{4/b}$.

Remark 6 *Therefore,* $\{a_0\}$ *is a collapsed gap of* σ_b *if, and only if,* $\{2a/b\}$ *is a collapsed gap of* $\sigma_{4/b}$.

Using duality, localization and reducibility to prevent collapsed gaps (Ince's argument revisited)

- For any $a, b, \omega \in \mathbb{R}$
$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z}. \tag{3}$$
cannot have more than one solution belonging to $l^2(\mathbb{Z})$.

- Assume $H_{b,0}$ is pure point. Then the set of point eigenvalues $\sigma_{pp}(H_{b,0})$ is dense in σ_b.

- Let a be a point eigenvalue of $\sigma_{pp}(H_{b,0})$ with eigenvector ψ.

- By duality, the Harper equation for $(2a/b, 4/b)$ has an quasi-periodic solution, $\tilde{\psi}$, with frequency ω.

- If $\{2a/b\}$ was a collapsed gap of $\sigma_{4/b}$ then the Harper equation would have another linearly quasi-periodic solution.

- By duality, (3) would have two solutions in $l^2(\mathbb{Z})$ ... CONTRADICTION!

Conclusion: Open spectral gaps are dense in σ_b (if $H_{b,0}$ is pure-point).

Key tool I : Non-perturbative localization

Theorem 7 (Jitomirskaya(1999)) *Let ω be Diophantine; i.e. be such that there exist positive constants c and $r > 1$ satisfying*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

for all $n \neq 0$. Define the set Φ of resonant phases as the set of those $\phi \in \mathbb{T}$ such that the relation

$$|\sin(\phi + \pi n\omega)| < \exp\left(-|n|^{\frac{1}{2r}}\right) \tag{4}$$

holds for infinitely many values of n. Then, if $\phi \notin \Phi$ and $|b| > 2$ the operator $H_{b,\phi}$ has only pure point spectrum with exponentially decaying eigenfunctions. Moreover, any of these eigenfunctions $(\psi_n)_{n\in\mathbb{Z}}$ satisfies that

$$\lim_{|n|\to\infty} \frac{\log\left(\psi_n^2 + \psi_{n+1}^2\right)}{2|n|} = -\log\left(\frac{|b|}{2}\right) = -\beta(b). \tag{5}$$

Remark 8 *For a similar result, see Bourgain & Goldstein (2000).*

Remark 9 *Using a result by Eliasson (1992), one only needs pure-point spectrum of $H_{b,\phi}$, for some $\phi \in \mathbb{T}$, to have Cantor spectrum.*

Some background on the Almost Mathieu Operator

- It was introduced to study electronic properties of solids: Harper (1955), Az'bel (1964), Rauth (1974), Hofstadter (1976), Aubry (1978), ...

- Duality was introduced by Aubry & André (1980) and proved rigorously by Avron & Simon (1983). This was completed by Jitomirskaya (1999).

- The Cantor spectrum of the A.M.O for irrational ω was conjectured by Az'bel. Kac (1981) conjectured that all gaps are open and Simon (1981) called it the Ten Martini Problem.

- For $b = 2$, Hofstadter gave numerical evidence that the measure of the spectrum is zero. This was proved by Last (1994) for most values of ω and by Avila & Krikorian (2003) for the remaining ω's.

- Bellissard & Simon (1982), proved that, for generic pairs of (ω, b), σ_b is a Cantor set.

- Sinaĭ (1987) proved that the spectrum is a Cantor set for Diophantine ω and $|b|$ small.

- Choi, Elliot & Yui (1990) proved that σ_b is a Cantor set for all $b \neq 0$ if ω is of some Liouville type.

Key tool II : reducibility of cocycles

A quasi-periodic skew-product on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$X_{n+1} = A(\theta_n)X_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega$$

is reducible to constant coefficients if there is a continuous $Z : \mathbb{T} \to SL(2,\mathbb{R})$ a Floquet matrix $B \in SL(2,\mathbb{R})$, such that

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B, \text{ holds for all } \theta \in \mathbb{T}. \tag{6}$$

If a quasi-periodic system is reducible, it can be conjugated to

$$Y_{n+1} = BY_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega,$$

and the dynamics are well-understood.

Remark 10 *If ω is rational, all skew-products are reducible to constant coefficients.*

Remark 11 *Even in this case, it may be necessary to "halve the frequency."*

Reducibility close to constant coefficients

Theorem 12 (Eliasson (1992)) *Assume that ω is Diophantine with constants c and r. Then there is a constant $C(c,r)$ such that, if $|b| < C(c,r)$ and rot(a,b) is either rational,*

$$\text{rot}(a,b) = \{k\omega\}/2, \text{ for some } k \in \mathbb{Z},$$

or Diophantine,

$$\left| \text{rot}(a,b) - \frac{\{k\omega\}}{2} \right| = \frac{1}{2} \left| k(a,b) - \{k\omega\} \right| \geq \frac{K}{|k|^\tau},$$

then

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - b\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \qquad (7)$$

on $\mathbb{R}^2 \times \mathbb{T}$ is reducible to constant coefficients, with Floquet matrix B. Moreover, if a is at an endpoint of a spectral gap of σ_b, then

$$\text{trace } B = \pm 2$$

and

$$B = \pm I \Leftrightarrow \text{the gap collapses.}$$

Conclusions

- We have seen that the Cantor spectrum is a direct consequence of non-perturbative localization, duality and reducibility.

- To prove that all spectral gaps are open a quantitative version of localization or reducibility is needed. In our case, this is acomplished by Eliasson's results on almost everywhere reducibility. This result, however, depends on the Diophantine condition on ω.

- The fact that the Almost Mathieu Operator is self-dual is very specific of this model.

- Non-perturbative localization has been extended to general potentials in one dimension.

- The combination of localization and reducibility is useful!