Alguns exemples i comentaris extres sobre el càlcul de residus:

20 (a) $f(z) = \dfrac{z^{1/2}}{z^3-4z^2+4z}$

Atenció: $z=0$ no és singularitat aïllada de $f(z)$; no podem definir $z^{1/2}$ holomorfa en un entorn de $z=0$ menys el zero. $z=2$ sí que és singularitat aïllada.

20 (b) $f(z) = \dfrac{e^{2z}}{z^2-z+1}$

$z^2-z+1=0 \iff z = \dfrac{1\pm\sqrt{3}i}{2}$ pols simples

$$\operatorname{Res}\left(f, \frac{1+\sqrt{3}i}{2}\right) = \lim_{z\to\frac{1+\sqrt{3}i}{2}} \left[\left(z-\frac{1+\sqrt{3}i}{2}\right)\frac{e^{2z}}{z^2-2z+1}\right] =$$

$$= \lim_{z\to\frac{1+\sqrt{3}i}{2}} \frac{e^{2z}}{z-\frac{1-\sqrt{3}i}{2}} = \frac{e^{1+\sqrt{3}i}}{\sqrt{3}i} = \frac{e}{\sqrt{3}}(\sin\sqrt{3} - i\cos\sqrt{3})$$

$$\operatorname{Res}\left(f, \frac{1-\sqrt{3}i}{2}\right) = \dots = \frac{e^{1-\sqrt{3}i}}{-\sqrt{3}i} = \overline{\frac{e^{1+\sqrt{3}i}}{\sqrt{3}i}} = \frac{e}{\sqrt{3}}(\sin\sqrt{3} + i\cos\sqrt{3})$$

21 (a) $f(z) = \dfrac{z-1}{z}e^{1/z}$ $\quad z=0$ singularitat essencial.

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

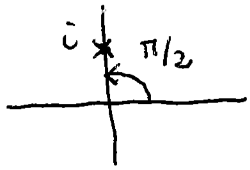
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

$$\frac{z-1}{z}e^{1/z} = \left(1-\frac{1}{z}\right)e^{1/z} = \left(1+\frac{1}{1!z}+\frac{1}{2!z^2}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{1!z}+\dots\right) =$$

$$= 1 + \left(1-\frac{1}{1!}\right)\frac{1}{z} + \left(\frac{1}{2!}-\frac{1}{3!}\right)\frac{1}{z^2} + \dots$$

$$\operatorname{Res}(f,0) = 1 - \frac{1}{1!} = 0.$$

21 (c) $f(z) = \dfrac{1}{\sinh(2\log z)}$ en $z = i$



$\underbrace{\log i}_{\text{determinació principal}} = \log|i| + i\arg(i) = i\pi/2$

$\sinh(2\log i) = \sinh(i\pi) = i\sin(\pi) = 0$

$\dfrac{d}{dz}(\sinh(2\log z))\Big|_{z=i} = \cosh(2\log z)\dfrac{2}{z}\Big|_{z=i} = \cosh(i\pi)\dfrac{2}{i} = \dfrac{2\cos\pi}{i} \neq$

$f(z)$ té pol simple en $z=i$

$\operatorname{Res}(f,i) = \lim_{z\to i} \dfrac{z-i}{\sinh(2\log z)} = \lim_{z\to i} \dfrac{1}{\cosh(2\log z)\frac{2}{z}} = -\dfrac{i}{2}$

21 (d) $f(z) = \dfrac{\sin z}{(z+i)^5}$ en $z = -i$

$\sin z = \sin(-i) + \dfrac{\cos(-i)}{1!}(z+i) - \dfrac{\sin(-i)}{2!}(z+i)^2 - \dfrac{\cos(-i)}{3!}(z+i)^3 + \dfrac{\sin(-i)}{4!}(z+i)^4 + \dots$

$\operatorname{Res}(f,-i) = \dfrac{\sin(-i)}{4} = -\dfrac{i\sinh(1)}{4!}$

22. $f(z) = \dfrac{1}{(2z+1)^3}$, $g(z) = \dfrac{\pi}{\sin(\pi z)}$

$\operatorname{Res}(f\cdot g, -1/2) = -\dfrac{\pi^3}{16} \leftarrow -1/2$ pol triple de $f\cdot g$

$\underset{\substack{\uparrow \\ \text{taylor en } -1/2}}{\sin(\pi z) =} -1 + \dfrac{\pi^2}{2}(z+1/2)^2 + \dots$

$\dfrac{\pi}{\sin(\pi z)} = \dfrac{\pi}{-1 + \frac{\pi^2}{2}(z+1/2)^2 + \dots} = \dfrac{-\pi}{1 - \frac{\pi^2}{2}(z+1/2)^2 + \dots} = -\pi\left(1 + \frac{\pi^2}{2}(z+1/2)^2 + \dots\right.$

$f(z)g(z) = \dfrac{1}{2^3(z+1/2)^3}(-\pi)\left(1 + \dfrac{\pi^2}{2}(z+1/2)^2 + \dots\right) =$

$= \dfrac{-\pi}{8}\left(\dfrac{1}{(z+1/2)^3} + \dfrac{\pi^2}{2}\dfrac{1}{z+1/2} + \dots\right)$