# Gorenstein Liaison of curves in $\mathbb{P}^4$

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## 1 Introduction.

Let $k$ be an algebraically closed field of characteristic zero, $S = k[X_0, X_1, X_2, X_3, X_4]$ and $\mathbb{P}^4 = Proj(S)$. By a curve we always mean a closed one-dimensional subscheme of $\mathbb{P}^4$ which is locally Cohen-Macaulay and equidimensional. The main purpose of this paper is to show that arithmetically Cohen-Macaulay curves $C \subset \mathbb{P}^4$ lying on a "general" arithmetically Cohen-Macaulay surface $X \subset \mathbb{P}^4$ with degree matrix $[u_{i,j}]$, $u_{i,j} > 0$, are glicci provided $16((KH)^2 - K^2H^2) - H^2[H^2 - K^2 + 8(1 + p_a)] \geq 0$; being $K$ the canonical divisor on X and $H$ the hyperplane section of $X$. We also give examples of arithmetically Cohen-Macaulay surfaces $X \subset \mathbb{P}^4$ verifying the above numerical condition.

The idea of using complete intersections to link varieties has been used for a long time, going back at least to work of Macaulay and Severi. Since them, many mathematicians have contributed to the development of liaison theory and we have a remarkable picture of the liaison theory and its applications in codimension 2. One naturally would like to carry out a program in higher codimension. In [KMMNP], the authors demonstrate that Gorenstein liaison is a natural generalization of the well-understood theory of CI-liaison codimension two cases. In particular, in [KMMNP]; Theorem 3.6 it is proved that every standard determinantal scheme $X \subset \mathbb{P}^n$ is glicci (i.e. $X$ belongs to the Gorenstein liaison class of a complete intersection), and it is posed the following question:

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**Question** ([KMMNP]): *In codimension three, is there only one Gorenstein liaison class containing arithmetically Cohen-Macaulay subschemes? or, equivalently, are all arithmetically Cohen-Macaulay subschemes glicci?*

In this paper, while we did not fully answer the question, we did make substantial progress. See Theorem 3.9 and Corollaries 3.10 and 3.11. As a main ingredient we use the fact that roughly speaking Gorenstein liaison is a theory about divisors on arithmetically Cohen-Macaulay schemes (See [KMMNP]) and the fact that the Picard group of a "general" arithmetically Cohen-Macaulay surface $X \subset \mathbb{P}^4$ is well known.

Next we outline the structure of the paper.

In section 2 we fix the notation and definitions needed in the sequel. At the beginning of section 3 we recall some results concerning arithmetically Cohen-Macaulay surfaces in $\mathbb{P}^4$ and curves lying on them. In particular, we collect some of the results obtained in [KMMNP] and a result of [L] describing the Picard group of general arithmetically Cohen-Macaulay surfaces in $\mathbb{P}^4$. Using these tools, in Theorem 3.9 we find a bound for the number of Gorenstein liaison classes containing arithmetically Cohen-Macaulay curves lying on a general arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$. This bound depends only on the invariants of the surface $X$, so we can compute it in specific cases. In particular, we get a large family of arithmetically Cohen-Macaulay surfaces in $\mathbb{P}^4$ where the bound is 1, which means that any arithmetically Cohen-Macaulay curve lying on them is glicci.

# 2 Definitions and basic facts.

In what follows we work over an algebraically closed field $k$ of characteristic 0. By $\mathbb{P}^n$ we denote the n-dimensional projective space over $k$ and by $R$ the polynomial ring $k[X_0, \ldots, X_n]$. For a subscheme $V$ of $\mathbb{P}^n$ we denote by $I_V$ its ideal sheaf and by $I(V)$ its saturated homogeneous ideal; note that $I(V) = H^0_*(I_V) := \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, I_V(t))$.

We recall that a codimension $c$ subscheme $X$ of $\mathbb{P}^n$ is *arithmetically Gorenstein* ( briefly *a.G.* ) if and only if its saturated homogeneous ideal, $I(X)$, has a minimal free resolution of the following kind:

$$0 \to R(-t) \to F_{c-1} \to \ldots \to F_1 \to F_0 \to I(X) \to 0.$$

In particular, $X$ is arithmetically Cohen-Macaulay ( briefly a.C.M. ). It is well known that in codimension two a.G. subschemes and complete intersection subschemes coincide. In higher codimension, any complete intersection subscheme is a.G. but not vice versa (indeed, a set of $n+2$ points in $\mathbb{P}^n$ in linear general position is a.G. but not complete intersection).

Now we collect the main definitions needed in this paper.

**Definition 2.1** (See also [KMMNP]; Definitions 2.3, 2.4 and 2.10) We say that two subschemes $V_1$ and $V_2$ are *Gorenstein linked*, or simply *G-linked*, by an a.G. scheme

$X$ if $I(X) \subset I(V_1) \cap I(V_2)$ and we have $I(X) : I(V_1) = I(V_2)$ and $I(X) : I(V_2) = I(V_1)$. We call *G-Liaison* the equivalence relation generated by G-links. If $X$ is a complete intersection, we say that $V_1$ and $V_2$ are *complete intersection linked*, or simply *CI-linked*. We call *CI-Liaison* the equivalence relation generated by CI-links. A subscheme $X \subset \mathbb{P}^n$ is said to be *licci* if it is in the CI-Liaison class of a complete intersection. Analogously, we say that a scheme $X \subset \mathbb{P}^n$ is *glicci* if it is in the Gorenstein Liaison class of a complete intersection.

We say that $V_1$ and $V_2$ are *CI-bilinked* (resp. *G-bilinked*) if $V_1$ is linked to $V_2$ in two steps by complete intersection schemes (resp. arithmetically Gorenstein schemes). For more information about Gorenstein liaison, see [M] and [KMMNP].

We are led to pose the following natural question:

*Do CI-Liaison and G-Liaison generate the same equivalence relation on codimension c subschemes of* $\mathbb{P}^n$ *?*

In codimension two the answer is yes, since complete intersections and a.G. schemes coincide. In higher codimension the answer is no. Indeed, a simple counterexample is the following: consider a set $X$ of four points in $\mathbb{P}^3$ in linear general position. By adding a sufficiently general fifth point we G-link $X$ to a single point. Therefore, $X$ is glicci. On the other hand, it follows from [HU]; Corollary 5.13 that $X$ is not licci.

The notion of using complete intersections to link varieties has been used for a long time, going back at least to work of Macaulay and Severi. The most complete picture in terms of Liaison theory and its applications has been in the codimension two case where arithmetically Gorenstein schemes and complete intersection schemes coincide. One naturally would like to develop a program in higher codimension and the results of [KMMNP] suggest that G-Liaison is a more natural approach to higher codimension.

# 3 Glicci curves on a.C.M. surfaces

In this section, we determine a huge family of a.C.M. surfaces $S \subset \mathbb{P}^4$ such that all a.C.M. curves $C$ lying on $S$ are glicci. Notice that this result drastically differs from the one obtained in [KMMNP]; Example 7.9. Indeed, in contrast to the fact that adding hyperplane sections does not preserve the CI-liaison class (see [KMMNP]; Corollary 7.5), we have

**Proposition 3.1** *Let* $X \subset \mathbb{P}^n$ *be a smooth a.C.M. subscheme and let* $C \subset X$ *be an effective divisor. Take any divisor* $C_1$ *in the linear system* $|C + tH|$ *being* $H$ *a hyperplane section of* $X$ *and* $t \in \mathbb{Z}$. *Then,* $C$ *and* $C_1$ *are G-bilinked. (Notice that if* $t = 0$ *then* $C$ *and* $C_1$ *are linearly equivalent.)*

*Proof.* See [KMMNP]; Corollary 5.13.

□

Proposition 3.1 motivates the following definition

**Definition 3.2** Let $X \subset \mathbb{P}^n$ be a smooth scheme. We say that an **effective divisor** $C$ on $X$ is **minimal** if there is no effective divisor in the linear system $|C - H|$ being $H$ a hyperplane section divisor of $X$.

Let us recall some facts on a.C.M. surfaces $X \subset \mathbb{P}^4$ and Lopez's Theorem which will allow us to bound the number of minimal a.C.M. curves on quite a lot a.C.M. surfaces $X \subset \mathbb{P}^4$ and, hence, the number of different G-Liaison classes containing a.C.M. curves $C$ on $X$. A surface $X \subset \mathbb{P}^4$ is said to be a.C.M. if it is homogeneous coordinate ring $S/I(X)$ is a Cohen-Macaulay ring. Given a such $X \subset \mathbb{P}^4$, there is a minimal free resolution

$$0 \longrightarrow \oplus_{i=1}^{n+1} S(-m_i) \xrightarrow{\phi} \oplus_{j=1}^{n+2} S(-d_j) \xrightarrow{\varphi} I(X) \longrightarrow 0.$$

where $n \geq 0$. We will assume $m_i \geq m_{i+1}$, $d_j \geq d_{j+1}$ and denote by $M_X = [A_{i,j}]$ and $(F_1, ..., F_{n+2})$ the matrices corresponding to $\phi$ and $\varphi$, respectively.

By the Hilbert-Burch theorem one can always assume that $F_j$ is the determinant of the matrix obtained from $M_X$ by removing the $j$-th column.

Set $u_{i,j} = m_i - d_j$ and $\delta M_X = [u_{i,j}]$. This last matrix is sometimes called the "degree matrix of $X$" since $u_{ij} = degA_{ij}$. Note that with the given ordering we have $u_{i+1,j} \leq u_{i,j} \leq u_{i,j+1}$.

**Terminology 3.3** To say that a statement holds for a general point of a projective variety $Y$ means that there exists a countable union $Z$ of proper subvarieties of $Y$ such that the statement holds for every $x \in Y \setminus Z$. In particular, we say that a statement holds for a *general* surface $X \subset \mathbb{P}^4$ with Hilbert polynomial $p(t)$ if the statement holds for a general point of some irreducible component of $Hilb_{p(t)}^{\mathbb{P}^4}$.

Unless otherwise specified the word general, when referred to elements of projective varieties, will have this meaning throughout this paper.

With the above notation we have:

**Theorem 3.4** *Let $X \subset \mathbb{P}^4$ be a general a.C.M. surface not being a complete intersection and such that $u_{i,j} > 0$ for all $i, j$. Then, three cases are possible for the Picard group of $X$:*
*(i) $Pic(X) \cong \mathbb{Z}^9$ and $X$ is a Castelnuovo surface, or*
*(ii) $Pic(X) \cong \mathbb{Z}^{11}$ and $X$ is a Bordiga surface, or*
*(iii) $Pic(X) \cong \mathbb{Z}^2$ if $X$ is none of the above.*

*Proof.* See [L]; Theorem III.4.2. □

**Remark 3.5** In the last case, Theorem 3.4(iii), $Pic(X)$ is generated by $H = \mathcal{O}_X(1)$ and $K$, being $K$ the canonical sheaf of $X$.

**Lemma 3.6** *Let $X \subset \mathbb{P}^4$ be a smooth general a.C.M. surface. Assume that either $X$ is a complete intersection or $X$ is rational. Then, any a.C.M. curve $C$ on $X$ is glicci.*

*Proof.* Indeed, either $X$ is rational and the result follows from [KMMNP]; Corollary 8.9, or $X$ is a complete intersection, $deg(X) > 4$ and $Pic(X) \cong \mathbb{Z} =< H >$. In this last case, the result follows from Proposition 3.1 and the fact that the hyperplane section $H$ of $X$ is an a.C.M. curve $C$ contained in $\mathbb{P}^3$, and according to Gaeta's Theorem [G], H is licci. □

From now on we restrict our attention to general, a.C.M. surfaces $X \subset \mathbb{P}^4$ which are neither rational, nor complete intersection. We will also assume that the degree matrix $[u_{i,j}]$ of $X$ verifies $u_{i,j} > 0$ for all i,j. According to Theorem 3.4, $Pic(X) \cong \mathbb{Z}H \oplus \mathbb{Z}K$. Set $d = H^2$ the degree of $X$, $\pi = \frac{H(H+K)}{2} + 1$ the sectional genus of $X$ and $p_a = \chi\mathcal{O}_X - 1$ the arithmetic genus of $X$. Define

$$(*) \quad m_0 := min\{0 \leq m \in \mathbb{Z} \mid H^2[H^2 - K^2 + 8(1+p_a)] \leq 4m^2((KH)^2 - K^2H^2)\}.$$

**Remark 3.7** Using the double point formula $2K^2 = d^2 - 5d - 10\pi + 12p_a + 22$, we can write $m_0$ in terms of the degree of $X$, the arithmetic genus of $X$ and the sectional genus of $X$:

$$m_0 = min\{0 \leq m \in \mathbb{Z} \mid 10\pi d - 6d + 7d^2 - d^3 + 4p_a d \leq$$

$$4m^2(8\pi^2 - 16\pi + 8 + 2\pi d - 14d + 7d^2 - d^3 - 12p_a d)\}.$$

Moreover, the invariants of $X$ can be written in terms of $m_i, i = 1, \dots, n+1$, and $d_j, j = 1, \dots, n+2$ in the following way:

$$d = (\sum_{i=1}^{n+1} m_i^2 - \sum_{j=1}^{n+2} d_j^2)/2$$

$$p_a = \sum_{i=1}^{n+1} \binom{m_i - 1}{4} - \sum_{j=1}^{n+2} \binom{d_j - 1}{4}$$

$$\pi = \sum_{i=1}^{n+1} \binom{m_i - 1}{3} - \sum_{j=1}^{n+2} \binom{d_j - 1}{3}.$$

Therefore, $m_0$ depends only on $m_i, i = 1, \dots, n+1$,and $d_j, j = 1, \dots, n+2$.

**Examples 3.8** (i) Let $X \subset \mathbb{P}^4$ be an a.C.M. surface defined by the maximal minors of a matrix $A$ with entries homogeneous forms of fixed degree $n$. $X$ has a graded minimal free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p-n)^{\frac{p+n}{n}-1} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p)^{\frac{p+n}{n}} \longrightarrow I_X \longrightarrow 0,$$

where $p \in \mathbb{N}$ is a multiple of $n$.

Computing the invariants of $X$, we get that $(KH)^2 - K^2H^2 = \frac{p^2(p+n)^2(p^2+np-2n^2)}{144}$ and $H^2[H^2 - K^2 + 8(1+p_a)] = \frac{p^2(p+n)^2(3p^2+3np+2n^2-8)}{48}$. Therefore, $m_0 = 2$ for all $p$ multiples of $n$, $p \geq 2n$, and for all $n \in \mathbb{N}$.

(ii) Let $X \subset \mathbb{P}^4$ be the a.C.M. surface defined by the maximal minors of a matrix $[A, B]$ where $A$ is an $n \times n$ matrix with linear entries and $B$ is a column with entries of degree $n$. Then $I_X$ has a graded minimal free resolution

$$0 \longrightarrow \mathcal{O}(-2n)^n \longrightarrow \mathcal{O}(-2n+1)^n \oplus \mathcal{O}(-n) \longrightarrow I_X \longrightarrow 0.$$

In this case, computing the invariants we get that $H^2[H^2 - K^2 + 8(1+p_a)] - 16((KH)^2 - K^2H^2) = \frac{n(n-1)(11n^3-124n^2+139n-14)}{144}$, which is $> 0$ for $n \geq 11$. Thus, for $n \geq 11$, $m_0 > 2$.

**Theorem 3.9** *Let $X \subset \mathbb{P}^4$ be a general a.C.M. surface with degree matrix $[u_{i,j}]$, $u_{i,j} > 0 \quad \forall i, j$. Then, there are at most $m_0 - 1$ G-liaison classes containing a.C.M. curves $C$ on $X$.*

*Proof.* Sine $u_{i,j} > 0$ for all i,j, we may assume that $X$ is smooth. Using [KMMNP]; Lemma 5.4, we have that any effective divisor in the linear system $nH - K, n \in \mathbb{Z}$ is an a.G. subscheme. So the effective divisors $aK + sH$ and $-(a+1)K + mH, a \geq 0$, are G-linked; in fact

$$aK + sH - (a+1)K + mH = -K + (s+m)H,$$

is a.G.

Thus, we just need to study the G-Liaison classes of $aK, a \geq 0$.

In the case $a = 0$, we have that any divisor in the linear system $\mid nH \mid$ is glicci because $H$ is an a.C.M. curve in $\mathbb{P}^3$. So we apply Gaeta's theorem to deduce that $H$ is licci and Proposition 3.1 to conclude that $nH$ is glicci.

Assume $a \geq 1$.

Let $C \subset X$ be any curve. Then, using the following exact sequence and the fact that $X$ is a.C.M.,

$$0 \longrightarrow I_{X,\mathbb{P}^4} \longrightarrow I_{C,\mathbb{P}^4} \longrightarrow I_{C,X} \longrightarrow 0$$

we have that $C$ is an a.C.M. curve if, and only if, $H^1(\mathcal{O}_X(-C+tH)) = 0, \forall t \in \mathbb{Z}$.

Thus, if $aK$ is an a.C.M. curve, we will have that

$$\chi(\mathcal{O}_X(-aK+tH)) = h^0(\mathcal{O}_X(-aK+tH)) + h^2(\mathcal{O}_X(-aK+tH)) \geq 0, \forall t \in \mathbb{Z}.$$

We compute $\chi(\mathcal{O}_X(-aK+tH))$ by the Riemman-Roch theorem:

$$\chi(\mathcal{O}_X(-aK+tH)) = \frac{H^2t^2 - (2a+1)KHt + a(a+1)K^2 + 2 + 2p_a}{2} \quad (1)$$

The equation $H^2t^2 - (2a+1)KHt + a(a+1)K^2 + 2 + 2p_a$, of degree two on $t$, has discriminant $D_a$,

$$D_a = 4((KH)^2 - K^2H^2)(a^2+a) + (KH)^2 - 8H^2(1+p_a).$$

By [H]; Exercise V.1.9 $(KH)^2 - K^2H^2 \geq 0$. If $(KH)^2 - K^2H^2 = 0$ then $H$ and $K$ are numerically equivalent. Thus $(KH)^2 - K^2H^2 > 0$ and for $a >> 0$ we will have $D_a \geq (H^2)^2 > 0$. This implies that the equation (1) on $t$ has two real solutions $t_0, t_1 \in \mathbb{R}$, $t_0 > t_1$ and $t_0 - t_1 \geq 1$ since $t_0 - t_1 = \frac{\sqrt{D_a}}{H^2}$.

Therefore, for $a >> 0$ there exists $t \in \mathbb{Z}$ such that $\chi(\mathcal{O}_X(-aK+tH)) \leq 0$. But, in fact, this is verified for any $a \geq m_0$. Indeed,

$D_a - (H^2)^2$ is an equation of order two on $a$ with discriminant

$$\Delta = 16((KH)^2 - K^2H^2)[((KH)^2 - K^2H^2) - (KH)^2 + 8H^2(1+p_a) + (H^2)^2].$$

If $\Delta \leq 0$, then any $a \geq 0$ verifies $D_a - (H^2)^2 \geq 0$.

If $\Delta > 0$, the solutions of $D_a - (H^2)^2$ are $-\frac{1}{2} \pm \frac{\sqrt{\Delta}}{8((KH)^2-K^2H^2)}$. Using this notation we have

$$m_0 = min\{0 \leq m \in \mathbb{Z} \mid \Delta \leq 64m^2((KH)^2 - K^2H^2)^2\},$$

so $m_0$ is the minimum integer $m$ such that $m \geq \frac{\sqrt{\Delta}}{8((KH)^2-K^2H^2)}$.

Thus, if $a \geq m_0$, we will have $a \geq -\frac{1}{2} + \frac{\sqrt{\Delta}}{8((KH)^2-K^2H^2)}$ and $D_a \geq (H^2)^2$, so there exists $t \in \mathbb{Z}$ such that $\chi(\mathcal{O}(-aK+tH)) < 0$. This means that $aK$ cannot be an a.C.M. curve for $a \geq m_0$.

Therefore the only G-Liaison classes which may contain a.C.M. curves are those determined by $aK$ with $0 \leq a \leq m_0 - 1$. We will know check that the ones determined by $H$ and $K$ coincide. In fact, we know that $H$ is licci (Indeed, $H$ is an a.C.M. curve contained in $\mathbb{P}^3$ and, by Gaeta's Theorem, $H$ is licci). Therefore, any effective divisor in the linear system $\mid nH \mid$ is glicci. Now we prove that also any effective divisor in the linear system $\mid K + lH \mid$ is glicci:

Let $L$ be the $(n+1) \times (n+2)$ matrix defining the surface $X$ and let $A = [L, M]$ be the matrix obtained adding to $L$ a column $M$. Thus $A$ defines a codimension 3 standard determinantal scheme $D \subset X \subset \mathbb{P}^4$. By [KMMNP]; Theorem 3.6, $D$ is glicci. Moreover, $\mathcal{O}_X(D) \cong \omega_X(t)$ for some $t \in \mathbb{Z}$, i.e., $D \in |K + tH|$ (see [KMMNP]; Proposition 10.7). Hence, $K$ and $D$ are G-bilinked (Proposition 3.1) so $K$ is glicci and is in the same G-Liaison class as $H$.

Therefore the number of G-Liaison classes containing a.C.M. curves on $X$ is at most $m_0 - 1$.

□

**Corollary 3.10** *Using the notation above, let $X \subset \mathbb{P}^4$ be a general a.C.M. surface with a graded minimal free resolution*

$$0 \longrightarrow \oplus_{i=1}^{n+1} \mathcal{O}(-m_i) \longrightarrow \oplus_{j=1}^{n+2} \mathcal{O}(-d_j) \longrightarrow I_X \longrightarrow 0.$$

*Assume that $m_0 = 2$ and $m_i - d_j > 0 \quad \forall i, j$. Then every a.C.M. curve $C \subset X$ is glicci.*

*Proof.* It follows directly from Theorem 3.9.

□

**Corollary 3.11** *Let $p \in \mathbb{N}$ be a multiple of $n \in \mathbb{N}$ and let $X_{p,n} \subset \mathbb{P}^4$ be a general a.C.M. surface with a graded minimal free resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p-n)^{\frac{p+n}{n}-1} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-p)^{\frac{p+n}{n}} \longrightarrow I_{X_{p,n}} \longrightarrow 0.$$

*Then, $m_0 = 2$, $\forall\ p$ and every a.C.M. curve $C \subset X_{p,n}$ is glicci.*

*Proof.* We may assume that $p \geq 2n$, because in the case $p = n$ $X_{p,n}$ is a complete intersection, and we may assume that $Pic(X_{p,n}) \cong \mathbb{Z}H \oplus \mathbb{Z}K$ (Theorem 3.4 and Remark 3.5).

As we have seen in Example 3.8(i), $m_0 = 2$ for all $p$ multiple of $n$, so we conclude by corollary 3.10. □

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