

COMPACT KÄHLER MANIFOLDS WITH ELLIPTIC HOMOTOPY TYPE

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ABSTRACT. Simply connected compact Kähler manifolds of dimension up to three with elliptic homotopy type are characterized in terms of their Hodge diamonds. For surfaces there are only two possibilities, namely $h^{1,1} \leq 2$ with $h^{p,q} = 0$ for $p \neq q$. For threefolds, there are three possibilities, namely $h^{1,1} \leq 3$ with $h^{p,q} = 0$ for $p \neq q$. This characterization in terms of the Hodge diamonds is applied to explicitly classify the simply connected Kähler surfaces and Fano threefolds with elliptic homotopy type.

1. INTRODUCTION

A finite, simply connected CW-complex has *elliptic homotopy type* if the total rank of its homotopy groups $\sum_{i \geq 2} \dim \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite.

A simply connected compact homogeneous manifold has elliptic homotopy type. From the homotopy-theoretic point of view simply connected manifolds with elliptic homotopy type constitute an extension of the class of 1-connected homogeneous manifolds (this is discussed in Section 2).

A manifold X with elliptic homotopy type has nice homotopical properties, such as:

- The loop space ΩX has Betti numbers

$$b_n(\Omega X) = O(n^r),$$

with $r = \dim \pi_{\text{odd}}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and, after localizing at finitely many primes, becomes homotopically equivalent to a product of spheres.

- Sullivan's minimal model for X turns into a finitely generated commutative differential graded algebra, simplifying its presentation as an algebraic scheme in algebro-geometric homotopy theory (see [13] and references therein).
- Due to the elliptic-hyperbolic alternative (see Theorem 2.1), the description of the homotopy groups of a manifold which is not of elliptic homotopy type becomes much more complex.

Our objective in this work is to identify the simply connected compact Kähler manifolds, of dimension up to three, that have elliptic homotopy type.

For dimension two, we classify them modulo a well-known open question in surface theory.

Theorem 1.1. *A 1-connected compact complex analytic surface has elliptic homotopy type if and only if it belongs to the following list:*

2000 *Mathematics Subject Classification.* 55P62, 14P25.

Key words and phrases. Rational homotopy, Kähler manifold, elliptic homotopy type, projective manifold.

- (i) the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$,
- (ii) Hirzebruch surfaces $\mathbb{S}_h = \mathbb{P}_{\mathbb{P}_{\mathbb{C}}^1}(\mathcal{O} \oplus \mathcal{O}(h))$, for $h \geq 0$, and
- (iii) 1-connected general type surfaces X with $q(X) = p_g(X) = 0$, $K_X^2 = 8$ and $c_2(X) = 4$.

The surfaces of type (iii) in Theorem 1.1 are called simply connected *fake quadrics*. The open question mentioned earlier is whether simply connected fake quadrics actually exist (see Remark 3.1). A simply connected fake quadric, if it exists, is homeomorphic to either the quadric $\mathbb{S}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ or the Hirzebruch surface \mathbb{S}_1 (the blow-up of $\mathbb{P}_{\mathbb{C}}^2$ at a point); see Remark 3.1. Therefore, Theorem 1.1 has the following corollary.

Corollary 1.2. *A 1-connected compact Kähler surface has elliptic homotopy type if and only if it is homeomorphic to either $\mathbb{P}_{\mathbb{C}}^2$ or to a Hirzebruch surface $\mathbb{S}_h = \mathbb{P}_{\mathbb{P}_{\mathbb{C}}^1}(\mathcal{O} \oplus \mathcal{O}(h))$ for some $h \geq 0$.*

Next we classify the compact Kähler threefolds with elliptic homotopy type in terms of the Hodge diamond.

Theorem 1.3. *A 1-connected compact Kähler threefold has elliptic homotopy type if and only if its Hodge diamond is one of the following:*

$$\begin{array}{c}
 \begin{array}{cccc}
 & & 1 & \\
 & 0 & & 0 \\
 0 & 0 & 1 & 0 \\
 (a) \quad 0 & 0 & 0 & 0 \\
 & 0 & 1 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}
 \qquad
 \begin{array}{cccc}
 & & 1 & \\
 & 0 & & 0 \\
 0 & 0 & 2 & 0 \\
 (b) \quad 0 & 0 & 0 & 0 \\
 & 0 & 2 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & & 1 & \\
 & & 0 & & 0 \\
 & 0 & 3 & & 0 \\
 (c) \quad 0 & 0 & 0 & 0 & 0 \\
 & 0 & 3 & & 0 \\
 & 0 & 0 & & \\
 & & 1 & &
 \end{array}
 \end{array}$$

A consequence of Theorems 1.1, 1.3 is that compact Kähler surfaces and threefolds with elliptic homotopy type are projective. Moreover, all examples known to the authors of 1-connected compact Kähler threefolds with elliptic homotopy type are in fact rationally connected. In the special case of homogeneous manifolds, Borel and Remmert proved that 1-connectedness implies rationality [4]. These considerations, and the birational classification of rationally connected threefolds by Kollár, Miyaoka, Mori in [15], have motivated us to classify the 1-connected Fano threefolds that are of elliptic homotopy type. This classification is carried out in Corollary 4.8 by applying Theorem 1.3. Most of the 1-connected Fano threefolds with elliptic homotopy type are neither homogeneous spaces nor fibrations over lower dimensional manifolds with elliptic homotopy type.

The known examples also led to the following generalization of the earlier mentioned question whether simply connected fake quadrics exist:

Question 1.4. *Are there 1-connected compact Kähler manifolds with elliptic homotopy type that are not rationally connected?*

Our results are proved by applying the Friedlander–Halperin bounds (recalled in Theorem 2.7) and the related properties of the rational homotopy of finite CW-complexes with elliptic homotopy type. The rich topological structure of compact Kähler manifolds arising from Hodge theory constrains the subclass of elliptic homotopy types.

The definition of elliptic homotopy type may be extended from simply connected spaces to nilpotent spaces, but the homotopy properties become more complex in that context. The only nilpotent compact Kähler manifolds known to the authors, up to finite étale coverings, are of the form $X \times T$, where X is 1-connected and T is a complex torus. It is easy to see that $X \times T$ has elliptic homotopy type if and only if X has it.

Acknowledgements. We are grateful to Ingrid Bauer, Najmuddin Fakhruddin, Ignasi Mundet i Riera and Rita Pardini for useful discussions. We wish to thank TIFR and CSIC for hospitality that made this work possible.

2. THE ELLIPTIC–HYPERBOLIC DICHOTOMY FOR HOMOTOPY TYPES

The homotopy groups of simply connected finite CW-complexes show a marked dichotomy, established in [7], which we will recall (for its proof and wider discussion, see also [8, § 33]).

Theorem 2.1. (The elliptic–hyperbolic dichotomy.) *Let X be a finite, 1-connected CW-complex X . The homotopy groups of X satisfy one of the two mutually exclusive properties:*

- (i) $\sum_{i \geq 2} \dim \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} < \infty$.
- (ii) $\sum_{i=2}^k \dim \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} > C^k$ for all k large enough, where $C > 1$ and it depends only on X .

Definition 2.2. The CW-complex X has *elliptic homotopy type* if (i) holds. The CW-complex X has *hyperbolic homotopy type* if (ii) holds.

If X has elliptic homotopy type, then almost all of its homotopy groups are torsion, and the Sullivan’s minimal model of X is a finitely generated algebra determining the rational homotopy type of the space. In contrast, if X has hyperbolic homotopy type then the Sullivan’s minimal model of X is a graded algebra, and the number of generators grow exponentially with the degree.

For any field k of characteristic zero, the k -homotopy groups of a 1-connected finite CW-complex X , and the Sullivan’s minimal model encoding the k -homotopy type of X , may be obtained from the \mathbb{Q} -homotopy groups and minimal model by the extension of scalars from \mathbb{Q} to k (see [21]). The same property of extension of scalars holds for cohomology algebras. So we will choose the coefficient field between \mathbb{Q} and \mathbb{R} according to convenience, and will say that X has *elliptic homotopy type* or *hyperbolic homotopy type* without any reference to the base field.

We start by presenting examples of manifolds with elliptic homotopy type. The first basic examples are:

Example 2.3. All 1-connected Lie groups and H -spaces of finite type have elliptic homotopy type.

Lemma 2.4. *Let $X \rightarrow B$ be a topologically locally trivial fibration, with fiber F , such that F , X and B are all 1-connected finite CW-complexes. If any two of them have elliptic homotopy type, then the third one also has elliptic homotopy type.*

Proof. Consider the associated long exact sequence of homotopy groups

$$\dots \rightarrow \pi_d(F) \rightarrow \pi_d(X) \rightarrow \pi_d(B) \rightarrow \dots$$

This exact sequence remains exact after tensoring with \mathbb{Q} . Hence the lemma follows. \square

This leads to the second set of basic examples.

Example 2.5. A 1-connected homogeneous manifold $X = G/H$, where H is a closed subgroup of a Lie group G , has elliptic homotopy type. It is not necessary that G and H be 1-connected. Since the action of π_1 on the higher homotopy groups is trivial for Lie groups, the homotopy exact sequence argument of Lemma 2.4 carries through in this case.

Example 2.6. Examples of compact Kähler manifolds with elliptic homotopy type provided by Lemma 2.4 and Example 2.5 include:

- (i) complex projective spaces $\mathbb{P}_{\mathbb{C}}^n = U(n+1)/(U(n) \times S^1)$.
- (ii) complex projective space bundles over a basis B of elliptic homotopy type, for instance, Hirzebruch surfaces

$$S_h = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1} \oplus \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(h)\right),$$

where h is a nonnegative integer.

We now recall a theorem of Friedlander and Halperin.

Theorem 2.7. (The Friedlander–Halperin bounds, [6].) *Let X be a 1-connected, finite CW-complex with elliptic homotopy type, and let m be the maximal degree d such that $H^d(X; \mathbb{Q}) \neq 0$. Select a basis $\{x_i\}_{i \in I}$ for the odd-degree homotopy $\pi_{\text{odd}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and also select a basis $\{y_j\}_{j \in J}$ for the even-degree homotopy $\pi_{\text{even}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then*

- (i) $\sum_{i \in I} |x_i| \leq 2m - 1$,
- (ii) $\sum_{j \in J} |y_j| \leq m$,
- (iii) $\sum_{i \in I} |x_i| - \sum_{j \in J} (|y_j| - 1) = m$,
- (iv) $\sum_{i \in I} |x_i| - \sum_{j \in J} |y_j| \geq 0$, and $e(X) \geq 0$.

(We have denoted by $|x|$ the degree of each homotopy generator x , while $e(X)$ is the topological Euler characteristic of X .)

The above inequality (iv) was proved in [10]. The rest were originally established in [6]. See [8, § 32] for a complete proof of it and related results.

The real homotopy groups of a manifold may be determined by computing the Sullivan minimal model of its commutative differential graded algebra (*cdga*) of global smooth differential forms. We note that this computation is easier for a closed Kähler manifold because such manifolds are formal (see [5]), and their Sullivan minimal model is that

of the cohomology algebra. The Friedlander–Halperin bounds in Theorem 2.7 and the initial steps in the computation of the Sullivan’s minimal model immediately impose some bounds on the Betti numbers of manifolds with elliptic homotopy type.

Corollary 2.8. *Let X be a 1-connected finite CW-complex of elliptic homotopy type. Then, $b_2(X) \leq m/2$, where $m = \dim X$.*

Proof. As X is simply connected, by Hurewicz’s theorem, there is an isomorphism $\pi_2(X) \cong H_2(X; \mathbb{Z})$. Therefore, it follows from inequality (ii) in Theorem 2.7 that

$$2b_2(X) = \sum_{y_j \in \text{basis of } \pi_2(X) \otimes_{\mathbb{Z}} \mathbb{Q}} |y_j| \leq \sum_{j \in J} |y_j| \leq m$$

completing the proof. □

Likewise, the following bound on the third Betti number of any 1-connected finite CW-complex X of elliptic homotopy type can be deduced from Theorem 2.7:

$$b_3(X) + \dim \ker \left(S^2 H^2(X; \mathbb{Q}) \xrightarrow{\cup} H^4(X; \mathbb{Q}) \right) \leq (2 \dim X - 1)/3$$

(S^j is the j -th symmetric product). We will prove a sharper bound for b_3 of closed symplectic manifolds with elliptic homotopy type (see Proposition 2.14). For that purpose, we will need another homotopical invariant of CW-complexes.

Definition 2.9. Let X be a connected finite CW-complex.

The *Lusternik–Schnirelmann category* of X , denoted $\text{cat}(X)$, is the least integer m such that X can be covered by $m + 1$ open subsets each contractible in X .

The *rational Lusternik–Schnirelmann category* of X , denoted $\text{cat}_0(X)$, is the least integer m such that there exists Y rationally homotopy equivalent to X with $\text{cat}(Y) = m$.

Some properties of the Lusternik–Schnirelmann category are listed below (see Proposition 27.5, Proposition 27.14 and § 28 in [8] for their proof).

Proposition 2.10. *Let X be a connected finite CW-complex.*

- (i) *The inequality $\text{cat}_0(X) \leq \text{cat}(X)$ holds.*
- (ii) *For any r -connected CW-complex X of dimension m ($r \geq 0$), the inequality $\text{cat}(X) \leq m/(r + 1)$ holds.*
- (iii) *The inequality*

$$\text{cup-length}(X) \leq \text{cat}_0(X)$$

holds, where cup-length(X) is the largest integer p such that there exists a product $\alpha_1 \cup \dots \cup \alpha_p \neq 0$ with $\alpha_i \in \bigoplus_{j>0} H^j(X; \mathbb{Q})$.

Using Proposition 2.10, we get the following property of symplectic manifolds (an equivalent version of it is proved in [22]).

Lemma 2.11. *Let (X, ω) be a 1-connected compact symplectic manifold with $\dim X = 2n$. Then,*

$$\text{cat}_0(X) = \text{cat}(X) = n.$$

Proof. We may perturb the original symplectic form ω to replace it by a symplectic form $\tilde{\omega}$ on X arbitrarily close to ω such that $[\tilde{\omega}] \in H^2(X; \mathbb{Q})$.

The inequalities in Proposition 2.10 yield

$$n \leq \text{cup-length}(X) \leq \text{cat}_0(X) \leq \text{cat}(X) \leq \frac{2n}{2} = n$$

completing the proof. \square

We now recall another property of CW-complexes that is a natural continuation of Theorem 2.7 (see [8, § 32]):

Proposition 2.12. (Friedlander–Halperin.) *If X is a 1-connected finite CW-complex with elliptic homotopy type, then*

$$\dim \pi_{\text{odd}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \text{cat}_0(X).$$

The following is an immediate consequence of Proposition 2.12 and Lemma 2.11.

Corollary 2.13. *Let X be a 1-connected compact symplectic manifold with elliptic homotopy type. Then $\dim \pi_{\text{odd}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \frac{\dim X}{2}$.*

Corollary 2.13 leads to the following bound on b_3 of Kähler and, more generally, symplectic manifolds.

Proposition 2.14. *Let X be a 1-connected compact symplectic manifold of dimension $2n$, and let r be the dimension of the kernel of the cup product map*

$$\cup : S^2 H^2(X; \mathbb{Q}) \longrightarrow H^4(X; \mathbb{Q}).$$

Assume that X has elliptic homotopy type. Then $b_3(X) \leq n - r$.

Proof. Lemma 2.11 implies that $\text{cat}_0(X) = \text{cat}(X) = n$. Therefore, by Corollary 2.13,

$$(2.1) \quad \dim \pi_3(X) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \dim \pi_{\text{odd}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \leq n.$$

The second stage in the computation of the Sullivan’s minimal model for X by induction on cohomology degree (see [9, Ch. IX]) shows that

$$(2.2) \quad \text{Hom}(\pi_3(X), \mathbb{Q}) \cong H^3(X; \mathbb{Q}) \oplus \text{kernel} \left(S^2 H^2(X; \mathbb{Q}) \xrightarrow{\cup} H^4(X; \mathbb{Q}) \right).$$

The proposition follows from (2.1) and (2.2). \square

3. COMPACT COMPLEX SURFACES WITH ELLIPTIC HOMOTOPY TYPE

In this section, 1-connected compact complex surfaces with elliptic homotopy type are investigated.

Theorem 1.1. *A 1-connected compact complex analytic surface has elliptic homotopy type if and only if it belongs to the following list:*

- (i) the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$,
- (ii) Hirzebruch surfaces $\mathbb{S}_h = \mathbb{P}_{\mathbb{P}_{\mathbb{C}}^1}(\mathcal{O} \oplus \mathcal{O}(h))$, where $h \geq 0$, and
- (iii) 1-connected surfaces of general type X with $q(X) = p_g(X) = 0$, $K_X^2 = 8$ and $c_2(X) = 4$.

Before proving the theorem, we make some remarks on its statement.

Remark 3.1. Projective surfaces X of general type with $q(X) = p_g(X) = 0$, $K_X^2 = 8$ and $c_2(X) = 4$ are commonly referred to as *fake quadrics*. Hirzebruch asked whether 1-connected fake quadrics exist. This question remains open. By Freedman's theorem (see [14, III, § 2]), any simply connected fake quadric is either homeomorphic to the Hirzebruch surface \mathbb{S}_1 or to the quadric $\mathbb{S}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$.

The bicanonical map $\Phi_{|2K|}$ of a fake quadric must be of degree 1 or 2. All fake quadrics with $\deg \Phi_{|2K|} = 2$ have been classified by M. Mendes Lopes and R. Pardini (see [16]), and each one of them has nontrivial fundamental group. Many fake quadrics with bicanonical map of degree one have been found by Bauer, Catanese, Grunewald and Pignatelli [2]. They are all uniformized by the bidisk, and have infinite fundamental group.

Now we will prove Theorem 1.1.

Proof. All 1-connected complex analytic surfaces admit Kähler metrics (see [1, Ch. VI.1]). Let X be a 1-connected Kähler surface.

The projective plane and the Hirzebruch surfaces have elliptic homotopy type (see Example 2.6). As we noted in Remark 3.1, any simply connected fake quadric is homeomorphic to a Hirzebruch surface. So if they exist, then they will have elliptic homotopy type as well.

To check that there are no other 1-connected Kähler surfaces with elliptic homotopy type, consider the Hodge numbers of any simply connected Kähler surface X with elliptic homotopy type. As X is 1-connected, we have $H^1(X; \mathbb{Q}) = 0$; so $H^3(X; \mathbb{Q}) = 0$ by Poincaré duality. By Corollary 2.8 we know that $b_2(X) \leq 2$. As $h^{1,1}(X) \geq 1$ and $h^{2,0}(X) = h^{0,2}(X)$, the only possibilities for the Hodge diamond of X are

$$\begin{array}{ccccc}
 & & 1 & & 1 \\
 & 0 & 0 & & 0 & 0 \\
 \text{(a)} & 0 & 1 & 0 & \text{(b)} & 0 & 2 & 0 \\
 & 0 & 0 & & 0 & 0 & & \\
 & & 1 & & & & 1 &
 \end{array}$$

If X has the Hodge diamond (a), then using the condition that X is simply connected, a theorem of Yau implies that X is the complex projective plane (see [1, § 5, Theorem 1.1]).

Let X be a simply connected Kähler surface possessing the Hodge diamond (b). Therefore,

$$(3.1) \quad \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) = 1 \quad \text{and} \quad c_2(X) = 2 - 4q(X) + b_2(X) = 4.$$

Also, we have

$$(3.2) \quad c_1(X)^2 = 12 \cdot \chi(\mathcal{O}_X) - c_2(X) = 8.$$

As $c_1(X)^2 > 0$ and $q(X) = 0$, the surface X is either rational or of general type.

As $h^{1,1}(X) = 2$, if X is rational it must be a Hirzebruch surface \mathbb{S}_h for some $h \geq 0$.

If X is of general type, then $h^{1,1}(X) = 2$ implies that X is minimal. This is the case of simply connected fake quadrics. As explained in Remark 3.1, it is currently unknown whether they exist. \square

4. COMPACT KÄHLER THREEFOLDS WITH ELLIPTIC HOMOTOPY TYPE

Just as in the case of surfaces, we start by finding out the Hodge diamonds of compact Kähler threefolds with elliptic homotopy type. From Corollary 2.8 we know that $b_2(X) \leq 3$, while Proposition 2.14 implies that $b_3(X) \leq 3$. In fact, a stronger statement holds as shown by the following proposition.

Proposition 4.1. *Let (X, ω) be a compact 1-connected symplectic six dimensional manifold with elliptic homotopy type. Then $b_3(X) = 0$.*

Proof. Since X is 1-connected, we have $b_1(X) = 0$. By Poincaré duality, $b_5(X) = 0$, while $b_4(X) = b_2(X)$. Since the cohomology classes represented by ω and ω^2 are nontrivial, the topological Euler characteristic $e(X)$ of X admits the bound

$$e(X) = 2 + 2b_2(X) - b_3(X) \geq 4 - b_3(X).$$

We noted above that $b_3(X) \leq 3$. Hence $e(X) > 0$. Halperin proved that if X has elliptic homotopy type, and $e(X) > 0$, then all odd Betti numbers of X vanish (see [10, p. 175, Theorem 1'(3)]). This completes the proof of the proposition. \square

Proposition 4.1 and Hodge theory immediately yield the following.

Corollary 4.2. *If X is a 1-connected compact Kähler threefold with elliptic homotopy type, then the Hodge diamond of X is one of the following four:*

$$\begin{array}{cc}
 \begin{array}{cccc}
 & & 1 & \\
 & & 0 & 0 \\
 & 0 & 1 & 0 \\
 (a) & 0 & 0 & 0 & 0 \\
 & & 0 & 1 & 0 \\
 & & & 0 & 0 \\
 & & & & 1 \\
 & & & & 1 \\
 & & & & 0 & 0 \\
 & 0 & 3 & 0 & \\
 (c) & 0 & 0 & 0 & 0 \\
 & & 0 & 3 & 0 \\
 & & & 0 & 0 \\
 & & & & 1
 \end{array}
 &
 \begin{array}{cccc}
 & & & 1 \\
 & & & 0 & 0 \\
 & & 0 & 2 & 0 \\
 (b) & 0 & 0 & 0 & 0 \\
 & & 0 & 2 & 0 \\
 & & & 0 & 0 \\
 & & & & 1 \\
 & & & & 1 \\
 & & & 0 & 0 \\
 & 1 & 1 & 1 & \\
 (d) & 0 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 \\
 & & & 0 & 0 \\
 & & & & 1
 \end{array}
 \end{array}$$

The last diamond in the above list is ruled out by the following proposition.

Proposition 4.3. *There does not exist any simply connected compact Kähler threefold possessing the Hodge diamond (d) in Corollary 4.2.*

Proof. Let X be a 3-fold having the Hodge diamond (d). Fix a Kähler form ω on X . As $H^{1,1}(X)$ is generated by ω , we have that $c_1(X) = \lambda\omega \in H^2(X; \mathbb{R})$ for some $\lambda \in \mathbb{R}$.

First assume that $\lambda > 0$. Therefore, the anti-canonical line bundle $\det TX := \bigwedge^{\text{top}} TX$ is positive, so X is complex projective and Fano. But Fano manifolds have $h^{2,0} = 0$, contradicting the Hodge diamond (d).

Assume now that $\lambda = 0$. Hence $c_1(X) = 0$. Since X is also simply connected, the canonical line bundle K_X is trivial. Hence $h^{3,0} = 1$, which contradicts the Hodge diamond (d).

Lastly, assume that $\lambda < 0$. This implies that the canonical line bundle K_X is positive. Therefore, the Miyaoka–Yau inequality says that

$$\int_X (c_1^2(X) - 3c_2(X))\omega \leq 0$$

(see [17, p. 449, Theorem 1.1], [25]). Substituting $\omega = \frac{1}{\lambda}c_1(X)$ in the above inequality,

$$(4.1) \quad \int_X c_1^3(X) \geq 3 \int_X c_1(X)c_2(X).$$

But from the Hodge diamond (d) and the Hirzebruch–Riemann–Roch theorem we derive that

$$2 = \chi(X, \mathcal{O}_X) = \int_X \frac{1}{24}c_1(X)c_2(X).$$

Therefore, $3 \int_X c_1(X)c_2(X) = 144$, while $\int_X c_1^3(X) = \frac{1}{\lambda^3} \int_X \omega^3 < 0$. This contradicts the inequality in (4.1), and completes the proof of the proposition. \square

The following proposition, which is a converse to Corollary 4.2 and Proposition 4.3, completes the proof of Theorem 1.3.

Proposition 4.4. *If a 1-connected compact Kähler threefold has the Hodge diamond (a), (b) or (c) in the list given in Corollary 4.2, then it has elliptic homotopy type.*

Proof. For each of the diamonds (a), (b) and (c), we will find a presentation for the real cohomology algebra $H^*(X; \mathbb{R})$ of any compact Kähler threefold X realizing the diamond in question. From these presentations we will derive Sullivan’s minimal model and ellipticity by using a Koszul complex.

The diamond (a) is the easiest to study, because we know from Hodge theory that if ω is a Kähler form on X , then $\omega^k \neq 0 \in H^{k,k}(X)$ for $k = 1, 2, 3$. Thus the real cohomology algebra of X is $H^*(X) = S^*(y)/(y^4) \cong H^*(\mathbb{P}_{\mathbb{C}}^3)$. The formality of closed Kähler manifolds implies that such X is real homotopy-equivalent to the complex projective space. Hence X has elliptic type as pointed out in Example 2.6(i). Its only nontrivial real homotopy groups are $\pi_2(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ and $\pi_7(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ (see [9, XIII.A]).

Next we consider diamond (b). Choose a basis $\{y_1, y_2\}$ for $H^2(X; \mathbb{R})$ such that y_1 is the class of the Kähler form on X , and y_2 is primitive. By the Hard Lefschetz Theorem, the pair $\{y_1^2, y_1y_2\}$ is a basis of $H^4(X; \mathbb{R})$; moreover, y_1^3 is the generator of $H^6(X; \mathbb{R})$ with positive orientation, and $y_1^2y_2 = 0 \in H^6(X; \mathbb{R})$.

Therefore,

$$y_2^2 = \alpha y_1^2 + \beta y_1 y_2 \in H^4(X; \mathbb{R})$$

for some scalars $\alpha, \beta \in \mathbb{R}$. The class y_2 is real and primitive. From the signature of the Q -pairing (see [23, Theorem 6.32]),

$$Q(y_2, \bar{y}_2) = - \int_X y_1 y_2^2 > 0$$

and $y_1 y_2^2 = \alpha y_1^3 + \beta y_1^2 y_2 = \alpha y_1^3$. Hence we have $\alpha < 0$. Rescaling y_2 , we may further impose the condition on the selected basis $\{y_1, y_2\}$ of $H^2(X)$ that

$$y_2^2 = -y_1^2 + \beta y_1 y_2$$

with $\beta \in \mathbb{R}$.

We have also shown that the cohomology algebra $H^*(X; \mathbb{R})$ is generated by $H^2(X; \mathbb{R})$. In other words, $H^*(X; \mathbb{R})$ is a quotient $\mathbb{R}[y_1, y_2]/\mathcal{D}$ of the commutative polynomial ring generated by y_1 and y_2 by an ideal of relations that we denote \mathcal{D} . We will prove that the two already identified relations $p_1(y_1, y_2) = y_1^2 - \beta y_1 y_2 + y_2^2$ and $p_2(y_1, y_2) = y_1^2 y_2$ actually generate the ideal \mathcal{D} .

For any nonnegative integer k , let

$$(\mathbb{R}[y_1, y_2]/(p_1, p_2))_k \subset \mathbb{R}[y_1, y_2]/(p_1, p_2)$$

be the linear subspace spanned by homogeneous polynomials of degree k . Note that $(\mathbb{R}[y_1, y_2]/(p_1, p_2))_1$ has basis $\{y_1, y_2\}$, and $(\mathbb{R}[y_1, y_2]/(p_1, p_2))_2$ has basis $\{y_1^2, y_1 y_2\}$, so they are isomorphic to $H^2(X; \mathbb{R})$ and $H^4(X; \mathbb{R})$ respectively.

We point out now that $(\mathbb{R}[y_1, y_2]/(p_1, p_2))_3$ has basis y_1^3 . First note that

$$\dim(\mathbb{R}[y_1, y_2])_3 = 4$$

and $(p_1, p_2)_3$ is generated by $y_1 p_1, y_2 p_1$ and p_2 . It is readily checked that these three generators and y_1^3 form a basis of $(\mathbb{R}[y_1, y_2])_3$.

Now we will show that $(\mathbb{R}[y_1, y_2])_4 = (p_1, p_2)_4$. The obvious inclusions are

$$\begin{aligned} y_1^4 &= y_1^2 p_1 - (\beta y_1 + y_2) p_2 \\ y_1^3 y_2 &= y_1 p_2 \\ y_1^2 y_2^2 &= y_2 p_2 \\ y_1 y_2^3 &= y_1 y_2 p_1 - (y_1 + \beta y_2) p_2 \end{aligned}$$

and using them we have $y_2^4 = y_2^2 p_1 - y_2 p_2 - \beta y_1 y_2^3 \in (p_1, p_2)_4$.

Finally, for degrees $k > 4$,

$$(\mathbb{R}[y_1, y_2])_k = (\mathbb{R}[y_1, y_2])_{k-4} \cdot (\mathbb{R}[y_1, y_2])_4 = (\mathbb{R}[y_1, y_2])_{k-4} \cdot (p_1, p_2)_4.$$

So assigning degree 2 to the variables y_1, y_2 , we have an isomorphism of graded \mathbb{R} -algebras

$$\mathbb{R}[y_1, y_2]/(p_1, p_2) \cong H^*(X; \mathbb{R}).$$

The above presentation for the cohomology algebra of X allows us to define a cdga $M = S^*(y_1, y_2) \otimes_{\mathbb{R}} \wedge^*(x_1, x_2)$ with degrees

$$|y_1| = |y_2| = 2, |x_1| = 3, |x_2| = 5,$$

and boundaries $dy_1 = dy_2 = 0, dx_1 = p_1(y_1, y_2), dx_2 = p_2(y_1, y_2)$. It is equipped with a cdga morphism

$$\rho : M \longrightarrow H^*(X)$$

that sends y_1, y_2 to their namesake cohomology classes, and x_1, x_2 to zero.

The algebra M is a Sullivan minimal cdga, so if ρ is a quasi-isomorphism, then M is the minimal model of X and yields the real homotopy groups of X .

To establish the quasi-isomorphism we are seeking, note that M is a *pure Sullivan algebra* according to the definition of [8, § 32]:

- it is finitely generated, $M = S^*Q \otimes \wedge^*P$, with $Q = \langle y_1, y_2 \rangle$ (the even degree generators), and $P = \langle x_1, x_2 \rangle$ (the odd degree generators) both finite dimensional;
- $d(Q) = 0$;
- $d(P) \subset S^*Q$.

A pure Sullivan algebra has a filtration counting the number of odd degree generators in every monomial, and the boundary operator d has degree -1 for this filtration. In this way, our differential algebra becomes the total space of a homological complex

$$C_\bullet = S^*Q \otimes_{\mathbb{R}} \wedge^\bullet P,$$

given by

$$S^*Q \otimes_{\mathbb{R}} \wedge^2 P \xrightarrow{d} S^*Q \otimes_{\mathbb{R}} P \xrightarrow{d} S^*Q$$

which is in fact the Koszul complex for the elements $p_1, p_2 \in S^*Q \cong \mathbb{R}[y_1, y_2]$. The homology of this complex is therefore $H_\bullet(C_\bullet) \cong H^*(M)$; this isomorphism is not graded. There is a short exact sequence

$$0 \longrightarrow H_{>0}(C_\bullet) \longrightarrow H_\bullet(C_\bullet) \longrightarrow H^*(X) \longrightarrow 0$$

with the last term being given by isomorphisms $H_0(C_\bullet) \cong S^*Q/(p_1, p_2) \cong H^*(X)$.

It remains now to show that $H_{>0}(C_\bullet) = 0$. To prove this, we first note that the ideal

$$\mathcal{D} = (p_1, p_2)$$

has radical (y_1, y_2) in $S^*Q = \mathbb{R}[y_1, y_2]$ because of the inclusion of the fourth power $(y_1, y_2)^4 \subset (p_1, p_2)$. Whenever the ideal generated by m homogeneous elements p_1, \dots, p_m in a polynomial ring $k[y_1, \dots, y_m]$ has radical (y_1, \dots, y_m) , the elements (p_1, \dots, p_m) form a regular sequence of maximal length, and they yield an acyclic Koszul complex. The version of this property in our cdga setting is:

Proposition 4.5. [8, § 32.3]. *For a pure Sullivan algebra $M = S^*Q \otimes \wedge^*P$ and associated homological complex C_\bullet as above,*

$$\bigoplus_{j>0} H_j(C_\bullet) = 0$$

if and only if the generators p_1, \dots, p_m of \mathcal{D} form a regular sequence.

We conclude that M is indeed the minimal model of the 3-fold X , and its nontrivial real homotopy groups are $\pi_2(X) \otimes \mathbb{R} \cong \mathbb{R}^2$, $\pi_3(X) \otimes \mathbb{R} \cong \mathbb{R}$ and $\pi_5(X) \otimes \mathbb{R} \cong \mathbb{R}$.

The proof for Hodge diamond (c) employs the same argument used for (b), with one additional parameter:

Using the Hard Lefschetz decomposition on real cohomology $H^*(X) = H^*(X; \mathbb{R})$, choose a basis $\{y_1, y_2, y_3\}$ of $H^2(X)$ such that y_1 is the class of the Kähler form while y_2 and y_3 are primitive classes, meaning $y_1^2 y_2 = 0 = y_1^2 y_3$. The elements $\{y_1^2, y_1 y_2, y_1 y_3\}$ form a basis of $H^4(X)$, and y_1^3 generates $H^6(X)$.

As in the case of diamond (b), the algebra $H^*(X)$ is generated by $H^2(X)$, and hence $H^*(X)$ is a quotient of the polynomial ring $S^*H^2(X) \cong \mathbb{R}[y_1, y_2, y_3]$ by an ideal \mathcal{D} of boundaries.

The above basis of $H^4(X)$ yields quadratic elements in \mathcal{D} :

$$\begin{aligned} y_2^2 - \alpha_{11}y_1^2 - \alpha_{12}y_1y_2 - \alpha_{13}y_1y_3 \\ y_2y_3 - \alpha_{21}y_1^2 - \alpha_{22}y_1y_2 - \alpha_{23}y_1y_3 \\ y_3^2 - \alpha_{31}y_1^2 - \alpha_{32}y_1y_2 - \alpha_{33}y_1y_3 \end{aligned}$$

for some $\alpha_{ij} \in \mathbb{R}$. As we pointed out for diamond (b), the Q -pairing is symmetric, and it is negative definite on real primitive cohomology, so by a \mathbb{R} -linear change of basis on the primitive cohomology group $P^2(X) = \langle y_2, y_3 \rangle$ we obtain a simplified form of the above boundaries in \mathcal{D} :

$$\begin{aligned} p_1(y_1, y_2, y_3) &= y_2^2 + y_1^2 - \beta_{12}y_1y_2 - \beta_{13}y_1y_3 \\ p_2(y_1, y_2, y_3) &= y_2y_3 - \beta_{22}y_1y_2 - \beta_{23}y_1y_3 \\ p_3(y_1, y_2, y_3) &= y_3^2 + y_1^2 - \beta_{32}y_1y_2 - \beta_{33}y_1y_3 \end{aligned}$$

The cohomology algebra $H^*(X)$ satisfies the condition that

$$y_1^2y_2 = 0 = y_1^2y_3,$$

and hence $y_1^2y_2$ and $y_1^2y_3$ belong to \mathcal{D} . This fact yields the final simplification among the parameters.

Lemma 4.6. *With the selected basis for primitive cohomology $P^2(X)$, the set of quadratic boundaries \mathcal{D}_2 is spanned by the three elements*

$$\begin{aligned} p_1(y_1, y_2, y_3) &= y_2^2 + y_1^2 - \alpha y_1y_2 - \beta y_1y_3 \\ p_2(y_1, y_2, y_3) &= y_2y_3 - \beta y_1y_2 - \gamma y_1y_3 \\ p_3(y_1, y_2, y_3) &= y_3^2 + y_1^2 - \gamma y_1y_2 - \delta y_1y_3 \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Proof. First note that

$$y_1y_2y_3 = y_1p_2 + \beta_{22}y_1^2y_2 + \beta_{23}y_1^2y_3 \in \mathcal{D}$$

and then

$$\begin{aligned} y_3p_1 &= y_2^2y_3 - \beta_{12}y_1y_2y_3 - \beta_{13}y_1y_3^2 &\implies y_2^2y_3 &= \beta_{13}y_1y_3^2 \pmod{\mathcal{D}} \\ y_2p_2 &= y_2^2y_3 - \beta_{22}y_1y_2^2 - \beta_{23}y_1y_2y_3 &\implies y_2^2y_3 &= \beta_{22}y_1y_2^2 \pmod{\mathcal{D}} \\ y_3p_2 &= y_2y_3^2 - \beta_{22}y_1y_2y_3 - \beta_{23}y_1y_3^2 &\implies y_2y_3^2 &= \beta_{23}y_1y_3^2 \pmod{\mathcal{D}} \\ y_2p_3 &= y_1^2y_2 - \beta_{32}y_1y_2^2 - \beta_{33}y_1y_2y_3 + y_2y_3^2 &\implies y_2y_3^2 &= \beta_{32}y_1y_2^2 \pmod{\mathcal{D}} \end{aligned}$$

The basis $\{y_2, y_3\}$ of $P^2(X)$ has been so selected that the Q -pairing yields

$$y_1y_2^2 = -y_1^3 = y_1y_3^2.$$

This element is a generator of $H^6(X)$. Therefore the above identities for $y_2^2y_3, y_2y_3^2$ imply that $\beta_{13} = \beta_{22}$ and $\beta_{23} = \beta_{32}$.

In this way we have found three boundary elements that are quadratic in y_1, y_2, y_3 , and are \mathbb{R} -linearly independent. As the dimension of $H^4(X)$ is three, and it is generated by products of degree two classes, we conclude that $\{p_1, p_2, p_3\}$ is a basis of \mathcal{D}_2 . \square

Continuing with the proof of Proposition 4.4, we will check that the ideal

$$\tilde{\mathcal{D}} = (p_1, p_2, p_3) \subseteq \mathcal{D}$$

is in fact the boundary ideal \mathcal{D} .

It follows from Lemma 4.6 that $H^{2k}(X) \cong \mathbb{R}[y_1, y_2, y_3]_k / \tilde{\mathcal{D}}_k$ for $k = 1, 2$. It also follows from the identities among parameters in p_1, p_2, p_3 that

$$y_1^2 y_2, y_1^2 y_3, y_1 y_2 y_3 \in \tilde{\mathcal{D}}_3.$$

The choice of y_2 and y_3 as Q -orthogonal means that

$$y_1 y_2^2 = y_1 y_3^2 = -y_1^3 \in \mathbb{R}[y_1, y_2, y_3] / \tilde{\mathcal{D}}.$$

Therefore, y_1^3 generates $\mathbb{R}[y_1, y_2, y_3]_3 / \tilde{\mathcal{D}}_3$, and it is nontrivial in cohomology, so it is a generator of the quotient.

Consider $k = 4$. Since any element of $\mathbb{R}[y_1, y_2, y_3]_3$ is congruent, modulo $\tilde{\mathcal{D}}$, to λy_1^3 for some $\lambda \in \mathbb{R}$, it follows that any element of $\mathbb{R}[y_1, y_2, y_3]_4$ is congruent modulo $\tilde{\mathcal{D}}$ to a linear combination of y_1^4 , $y_1^3 y_2$ and $y_1^3 y_3$. The last two monomials are multiples of $y_1^2 y_2$ and $y_1^2 y_3$ respectively, and so they lie in $\tilde{\mathcal{D}}$. It is straight forward to verify that y_1^4 is a linear combination of $y_1^2 p_1$ and elements of $\tilde{\mathcal{D}}_4$. The conclusion is that $\mathbb{R}[y_1, y_2, y_3]_4 = \tilde{\mathcal{D}}_4 = \mathcal{D}_4$.

As was observed for diamond (b), for $k > 4$,

$$\mathbb{R}[y_1, y_2, y_3]_k = \mathbb{R}[y_1, y_2, y_3]_{k-4} \cdot \mathbb{R}[y_1, y_2, y_3]_4 = \mathbb{R}[y_1, y_2, y_3]_{k-4} \cdot \tilde{\mathcal{D}}_4 = \tilde{\mathcal{D}}_k.$$

Putting all the homogeneous components together, we have proved that $\mathcal{D} = \tilde{\mathcal{D}}$. In other words, $H^*(X) = \mathbb{R}[y_1, y_2, y_3] / (p_1, p_2, p_3)$.

Let $M = S^*V^2 \otimes \wedge^*V^3$ be the free cdga generated by $V^2 = \langle y_1, y_2, y_3 \rangle$ in degree two and $V^3 = \langle x_1, x_2, x_3 \rangle$ in degree three, with boundaries

$$dy_1 = dy_2 = dy_3 = 0 \quad \text{and} \quad dx_j = p_j \quad j = 1, 2, 3,$$

where $p_j \in \mathbb{R}[y_1, y_2, y_3] \cong S^*V^2$ are the generators of \mathcal{D} .

By its definition, the algebra M is endowed with a morphism

$$\rho : M \longrightarrow H^*(X).$$

We will prove that ρ is a quasi-isomorphism.

This is done as in the case of Hodge diamond (b): again M is a pure Sullivan algebra; the filtration by number of odd degree variables defines a Koszul complex structure on M , and there is a short exact sequence

$$0 \longrightarrow H_{>0}(C_\bullet) \longrightarrow H^*M \longrightarrow H^*(X) \longrightarrow 0.$$

As before, the radical of \mathcal{D} is (y_1, y_2, y_3) because of the inclusion of its fourth power $(y_1, y_2, y_3)^4 \subset \mathcal{D}$, thus the 3 generators p_1, p_2, p_3 form a regular sequence in $\mathbb{R}[y_1, y_2, y_3]$. Thus the Koszul complex C_\bullet has $H_{>0}(C_\bullet) = 0$, and M is the minimal model of $H^*(X)$. The only nontrivial real homotopy groups of X are $\pi_2(X) \otimes \mathbb{R} \cong \mathbb{R}^3$ and $\pi_3(X) \otimes \mathbb{R} \cong \mathbb{R}^3$. This completes the proof of Proposition 4.4. \square

Corollary 4.2, Proposition 4.3 and Proposition 4.4 together complete the proof of Theorem 1.3. Examples of 1-connected projective threefolds realizing the three possible Hodge diamonds are readily found using Lemma 2.4:

- Example 4.7.* (i) The projective space $\mathbb{P}_{\mathbb{C}}^3$ and the quadric threefold $Q_2 \subset \mathbb{P}_{\mathbb{C}}^4$ have Hodge diamond (a).
(ii) $\mathbb{P}_{\mathbb{C}}^1$ -bundles over $\mathbb{P}_{\mathbb{C}}^2$ and $\mathbb{P}_{\mathbb{C}}^2$ -bundles over $\mathbb{P}_{\mathbb{C}}^1$ have Hodge diamond (b).
(iii) $\mathbb{P}_{\mathbb{C}}^1$ -bundles over a Hirzebruch surface \mathbb{S}_h , and bundles of Hirzebruch surfaces over $\mathbb{P}_{\mathbb{C}}^1$ have Hodge diamond (c).

Finally, we can use the classification up to deformation of Fano threefolds by Iskovskih ([11], [12]) and Mori–Mukai ([18]) to determine which ones among them have elliptic homotopy type. This shows that, starting in dimension 3, not all projective manifolds with elliptic homotopy type are obtained by fibering homogeneous spaces.

The entries in the table of Fano threefolds at the end of [18] are henceforth referred to as “entries” without further clarification.

Corollary 4.8. *Let X be a Fano threefold with elliptic homotopy type.*

If X has Hodge diamond (a) in Theorem 1.3, then, up to deformations, X is one of the following:

- (i) $\mathbb{P}_{\mathbb{C}}^3$,
- (ii) the quadric $Q \subset \mathbb{P}_{\mathbb{C}}^4$,
- (iii) the Fano manifold $X_{22} \subset \mathbb{P}_{\mathbb{C}}^{13}$ with genus $g = 12$, (see [19], [20]).

If X has Hodge diamond (b) in Theorem 1.3, then, up to deformations, X is one of the following: entries 20, 21, 22, 24, 26, 27, 29, 30, 31, 32, 33, 34, 35 and 36 in the list of Fano 3-folds with $b_2(X) = 2$ given in [18].

If X has Hodge diamond (c) in Theorem 1.3, then, up to deformations, X is one of the following: entries 5, 8, 10, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30 and 31 in the list of Fano 3-folds with $b_2(X) = 3$ in [18].

Proof. As a Fano manifold X has no holomorphic forms of positive degree, it satisfies $b_2(X) = h^{1,1}(X)$, and X can have Hodge diamond (a), (b) or (c) in Theorem 1.3 if and only if $b_3(X) = 0$ and $b_2(X) = 1, 2$ or 3 respectively. Therefore it suffices to go over the classification of Fano threefolds and check the Betti numbers b_2 and b_3 . \square

Remark 4.9. Comparing the classification in Corollary 4.8 with the classification of homogeneous complex manifolds of dimension three in [24] we conclude that $\mathbb{P}_{\mathbb{C}}^3$ and the quadric threefold $Q \subset \mathbb{P}_{\mathbb{C}}^4$ are the only homogeneous spaces among the 1-connected Fano threefolds with elliptic homotopy type.

Other cases in the classification, starting with the Fano manifold X_{22} of genus 12, cannot be obtained either as fibrations over lower dimensional manifolds with elliptic homotopy type.

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