Comment on “Effects of quantized scalar fields in cosmological spacetimes with big rip singularities”

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There are two *non equivalent* ways to check if quantum effects in the context of semiclassical gravity can moderate or even cancel the final singularity appearing in a universe filled with dark energy:

The method followed in [1] is to introduce the classical Friedmann solution in the energy density of the quantum field, and to compare the result with the density of dark energy determined by the Friedmann equation.

The method followed in this comment is to solve directly the semiclassical equations.

The results obtained by either method are very different, leading to opposed conclusions.

The authors of [1] find that for a perfect fluid with state equation $p = \omega\rho$ and $\omega < -1$ (phantom fluid), considering realistic values of ω leads to a quantum field energy density that remains small compared to the dark energy density until the curvature reaches the Planck scale or higher, at which point the semiclassical approach stops being valid. The conclusion is that quantum effects do not affect significantly the expansion of the universe until the scalar curvature reaches the Planck scale.

In this comment we will show by numerical integration of the semiclassical equations that quantum effects modify drastically the expansion of the universe from an early point. We also present an analytic argument explaining why the method of [1] fails to detect this. The units employed are the same as in [1] ($c = \hbar = G = 1$).

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To obtain the dynamical equations in semiclassical gravity for a massless scalar conformally coupled field, one has to use the anomalous trace given by [2]

$$T_{vac} = \frac{1}{2880\pi^2}\left(\Box R + \frac{1}{2}\mathcal{G}\right), \tag{1}$$

with R the scalar curvature and $\mathcal{G} = -2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$ the Gauss-Bonnet curvature invariant, where we have used that the Weyl tensor vanishes in a FRW geometry to obtain this expression of $\mathcal{G}$.

In terms of the Hubble parameter equation (1) can be written as follows

$$T_{vac} = \frac{1}{480\pi^2}(\dddot{H} + 12H^2\dot{H} + 7H\ddot{H} + 4\dot{H}^2) + \frac{1}{240\pi^2}(H^4 + H^2\dot{H}), \tag{2}$$

thus, using the trace anomaly $T_{vac} = \rho_{vac} - 3p_{vac}$ and inserting (2) in the conservation equation $\dot{\rho}_{vac} + 3H(\rho_{vac} + p_{vac}) = 0$ one easily obtains the vacuum energy density [3]

$$\rho_{vac} = \frac{1}{480\pi^2}(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2) + \frac{1}{960\pi^2}H^4, \tag{3}$$

and then the semiclassical Friedmann equation is given by $H^2 = \frac{8\pi}{3}(\rho + \rho_{vac})$.

Finally, using the dimensionless variables $\bar{t} = H_+ t$, $\bar{H} = H/H_+$ $\bar{Y} = \dot{H}/H_+^2$ and $\bar{\rho} = \frac{8\pi\rho}{3H_+^2}$ with $H_+ = \sqrt{360\pi}$, the semiclassical Friedmann equation and the conservation equation can be written as the following differential system that we will integrate numerically

$$\left\{\begin{array}{lcc} \bar{H}' & = & \bar{Y} \\ \bar{Y}' & = & \frac{1}{2\bar{H}}\left(\bar{H}^2 - \bar{\rho} - 6\bar{H}^2\bar{Y} + \bar{Y}^2 - \bar{H}^4\right) \\ \bar{\rho}' & = & -3\bar{H}(1+\omega)\bar{\rho}, \end{array}\right. \tag{4}$$

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where $'$ denotes the derivative with respect the time $\bar{t}$.

In [1], from the classical Friedmann solution

$$H_F(t) = \frac{2}{3(1+\omega)}\frac{1}{t-t_s}, \tag{5}$$

where $t_s \equiv -\frac{2}{3H_0(1+\omega)}$, being $H_0 = H(0)$ the initial condition, the authors calculate the dark energy density

$$\rho_F = \frac{3H^2}{8\pi} = \frac{1}{6\pi(1+\omega)^2}\frac{1}{(t-t_s)^2}, \tag{6}$$

which in terms of the scalar curvature $R = 6(2H^2+\dot{H}) = \frac{4(1-3\omega)}{3(1+\omega)^2}\frac{1}{(t-t_s)^2}$ can be written as $\rho_F = \frac{R}{8\pi(1-3\omega)}$.

After this, the authors insert the classical Friedmann solution in (3) to obtain

$$\rho_{vac} = \frac{1}{34560\pi^2}\frac{27\omega^2+18\omega-5}{(1-3\omega)^2}R^2. \tag{7}$$

Then comparing equation (6) and (7), they deduce that for realistic values of ω, for example $\omega = -1.25$, when the scalar curvature is well bellow to the Planck scale ($R<1$), one has $\rho_{vac} \ll \rho_F$, and thus, they conclude that the quantum effects don't affect significantly the expansion of the universe until the spacetime curvature is of the order of the Planck scale where the semiclassical approximation breaks down, that is, semiclassical gravity does not milden or avoid the big rip singularity.

To prove analytically that their conclusions are wrong we perform the change of variable $\bar{p} \equiv \sqrt{|\bar{H}|}$ which gives the following semiclassical Friedmann equation

$$\frac{d}{d\bar{t}}\left((\bar{p}^{'})^2/2 + V(\bar{p})\right) = -3\epsilon p^2(\bar{p}^{'})^2 + \frac{3\epsilon}{8}(1+\omega))\bar{\rho}, \tag{8}$$

where $V(\bar{p}) = -\frac{1}{8}\left(\bar{p}^2(1-\frac{1}{3}\bar{p}^4) + \frac{\bar{\rho}}{\bar{p}^2}\right)$, and $\epsilon \equiv sign(\bar{H})$.

Note that this system is dissipative (anti-dissipative) in the region $\bar{H} > 0$ ($\bar{H} < 0$). The potential V (its picture appears in figure 3 of ref. [4]) has two critical points at $\bar{p}_\pm = \left(\frac{1\pm\sqrt{1-4\bar{\rho}}}{2}\right)^{1/4}$ ($\bar{p}_- < \bar{p}_+$). Then for $\bar{\rho} > 1/4$ there are no critical points, and the potential is strictly increasing from $-\infty$ to ∞. On the other hand, for $\bar{\rho} < 1/4$, the potential satisfies $V(0) = -\infty$, $V(\infty) = \infty$ and has a relative maximum at $\bar{p}_-$ and a relative minimum at $\bar{p}_+$ (a hollow). For very small values of $\bar{\rho}$ at $\bar{p}_-$ one has $\bar{H}^2 \cong \bar{\rho}$, that is, the system is close to the Friedmann phase, and at $\bar{p}_+$ one has $|\bar{H}| \cong 1$, that is, the system is close to the de Sitter phase.

Assuming that the system is in the expanding Friedmann phase at early times, i.e., it is in a point close to $\bar{p}_-$, namely $\bar{p}_F$, that satisfies $\bar{p}_F < \bar{p}_-$. Then, for realistic values of ω, the system immediately leaves the expanding Friedmann phase and rolls down either to the left or to right. In the former case (we have checked numerically that this happens for $\omega \geq -1.25$) it rolls down to to $\bar{p} = 0$ with a decreasing scalar curvature because $R = 4320\pi\bar{p}(1+\bar{p}')$. At $\bar{p} = 0$ the universe bounces and enters in a decreasing phase $\bar{H} < 0$, where the scalar curvature continues decreasing and arrives at Planck scales ($R = -1$) in a finite time (details of this behavior are given in [5]). This behavior is shown in figures $1, 2$.

In the other case (we have checked numerically that this happens for $\omega < -1.25$) the universe approaches an expanding de Sitter phase (the relative minimum $\bar{p}_+$) and the scalar curvature has values greater than the Planck scale, thus the semiclassical approximation breaks down. However, since $\bar{\rho}$ is an increasing function of time in the region $\bar{H} > 0$, the critical points will disappear and the potential will be an increasing function of $\bar{p}$. This means that the universe rolls down, once again, to $\bar{p} = 0$, and enters a decreasing phase $\bar{H} < 0$. This behavior is shown in figures $3, 4$.

In conclusion we have seen numerically that the correct behavior of the solutions differs from that obtained in [1] and in others papers (see for instance [6]) where authors used the first method described in the abstract to study the problem. More precisely, what we have seen numerically, in the context of semiclassical gravity, is that if at early times the universe is in the expanding Friedmann phase, quantum effects will modify drastically its expansion.

Finally, two remarks are in order: First, note that in this comment we have only studied the massless conformally coupled fields since for these the energy density is known to be completely specified in terms of the trace anomaly, in stark contrast to the massive case and/or non-conformally coupled case, where complicated state-dependent contributions in the quantum energy density appear and it seems very difficult to obtain a differential system like (4). However, from the results obtained in the conformally coupled case one may expect that the quantum effects modify the expansion of the universe and moderate the singularities. Those cases deserve future investigation.

Second, for very small values of $\bar{H}_0$ the solution of (4) remains for a long time close to the Friedmann phase (see figure 5). The solution of (4) remains in the Friedmann phase until $\bar{H}(t)$ reaches a sufficiently large value, such as 10^{-4} in figure 1, and at this moment its behavior changes drastically. The time interval for which $\bar{H}(t)$ is close to the Friedmann phase can be estimated from (5), and it has order $\bar{t} \sim -\frac{2}{3(1+\omega)\bar{H}_0}$. This behavior happens because for very small values of $\bar{H}_0$, quantum corrections can be considered as a small perturbation. However for larger values of $\bar{H}(t)$, even though the scalar curvature is well below to the Planck scale, quantum effects change drastically the expansion of the universe. In the present universe, the current value of Hubble's parameter is of the order $H_0 \sim 10^2 kms^{-1}Mpc^{-1}$ and, since $1s \sim 10^{43} t_{Pl}$, in Planck units one has $H_0 \sim 10^{-60} t_{Pl}^{-1}$. Thus the time remaining in the Friedmann phase is of the order $\bar{t} \sim 10^{62} \gg \bar{t}_{Pl} = \sqrt{360\pi}$ (in the units used in the comment $t_{Pl} = 1$).

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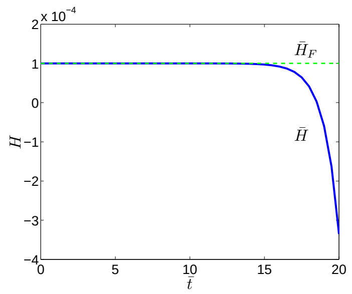

Figure 1: $\bar{H}(\bar{t})$ obtained by integration of (4) with initicial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and the classical Friedmann solution $\bar{H}_F(\bar{t}) = \frac{\bar{H}_0}{1+\frac{3(1+\omega)}{2}\bar{H}_0\bar{t}}$ in dimensionless variables ($\omega = -1.25, \bar{H}_0 = 10^{-4}$).

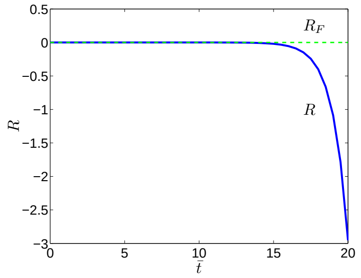

Figure 2: Scalar curvature $R(\bar{t})$ obtained by integration of (4) with initicial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and the classical Friedmann scalar curvature $R_F(\bar{t}) = 1080\pi(1-3\omega)\bar{H}_F^2(\bar{t})$ obtained by inserting in the scalar curvature the classical Friedmann solution ($\omega = -1.25, \bar{H}_0 = 10^{-4}$).

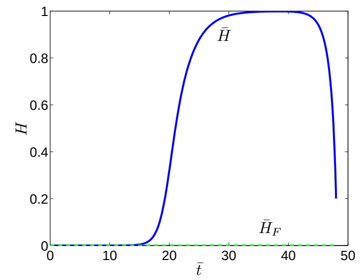

Figure 3: $\bar{H}(\bar{t})$ obtained by integration of (4) with initicial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and the classical Friedmann solution $\bar{H}_F(\bar{t}) = \frac{\bar{H}_0}{1+\frac{3(1+\omega)}{2}\bar{H}_0\bar{t}}$ in dimensionless variables ($\omega = -1.5, \bar{H}_0 = 10^{-4}$). Note that $\bar{H}$ initially grows, before going to negative values.

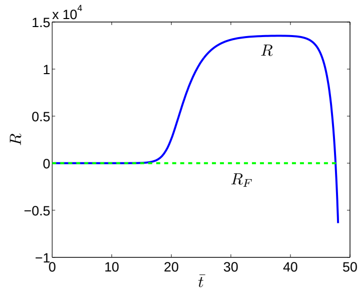

Figure 4: Scalar curvature $R(\bar{t})$ obtained by integration of (4) with initicial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and the classical Friedmann scalar curvature $R_F(\bar{t}) = 1080\pi(1-3\omega)\bar{H}_F^2(\bar{t})$ obtained by inserting in the scalar curvature the classical Friedmann solution ($\omega = -1.5, \bar{H}_0 = 10^{-4}$).

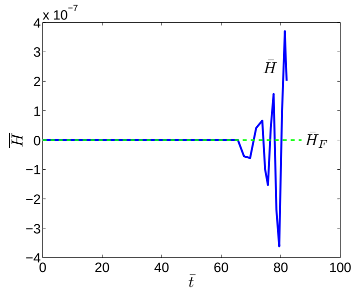

Figure 5: $\bar{H}(\bar{t})$ obtained by integration of (4) with initial conditions $\bar{H}_0 = 10^{-16}, Y_0 = -\frac{3}{2}(1+\omega)\bar{H}_0^2, \rho_0 = \bar{H}_0^2$ and the classical Friedmann solution $\bar{H}_F(\bar{t}) = \frac{\bar{H}_0}{1+\frac{3(1+\omega)}{2}\bar{H}_0\bar{t}}$ in dimensionless variables ($\omega = -1.25, \bar{H}_0 = 10^{-4}$). Note the agreement of both solutions up to $\bar{t} \approx 2\bar{t}_{Pl}$, and how they diverge afterwards.