On the fate of the phantom dark energy universe in semiclassical gravity

Jaume Haro[1,*], Jaume Amoros[1,†] and Emilio Elizalde[2,‡]
[1]*Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain*
[2]*Instituto de Ciencias del Espacio (CSIC) & Institut d'Estudis Espacials de Catalunya (IEEC/CSIC)*
Campus UAB, Facultat de Ciències, Torre C5-Parell-2a planta, 08193 Bellaterra (Barcelona) Spain

The fate of the phantom dark energy universe in semiclassical gravity is investigated. Quantum corrections coming from massless fields conformally coupled with gravity are considered, to see if they can lead to avoidance of the Big Rip singularity, which shows up in a flat Friedmann-Robertson-Walker universe, filled with phantom dark energy and modeled by an equation of state of the form $p = \omega\rho$ with $\omega < -1$. The dynamics of the model are discussed for all values of the two parameters, named $\alpha > 0$ and $\beta < 0$, which come from quantum corrections. It is concluded that, when $-1 < \frac{\beta}{3\alpha} < 0$, almost all solutions develop future singularities (the corresponding scale factor and energy density go down to zero in finite time). However, when $-1 > \frac{\beta}{3\alpha}$, almost all solutions describe a universe bouncing infinitely many times (an oscillating universe).

PACS numbers: 98.80.Qc, 04.62.+v, 04.20.Dw

I. INTRODUCTION

Studies of distant type Ia supernovae [1, 2] (and others) provide strong evidence that our universe is expanding in an accelerating way. To explain such acceleration one usually assumes the existence of phantom dark energy with a negative pressure. However, the classical solutions of general relativity for a Friedmann-Robertson-Walker (FRW) model containing phantom dark energy lead, in general, to future singularities [3–5] (Big Rip, future sudden singularities, etc.). Recently a good number of papers have been dealing with the possibility to avoid these future singularities, by using semiclassical gravity [5–11]. However, there is still lack of agreement concerning the final conclusions. Discrepancies occur, basically, because there appear to be two "non-equivalent" ways to analyze the problem. The first is to compute the energy density of the quantum field, by inserting in it the classical Friedmann solution and then comparing the result obtained with the phantom energy. This method seems to lead to the conclusion that, in general, quantum effects do not significantly affect the universe expansion [11]. The alternative way is trying to solve the semiclassical back-reaction equations or, if this fails, making a qualitative study in order to see the influence of que quantum fields on the fate of the universe. This second procedure shows that quantum corrections can drastically change the behavior of the classical solutions. It will be demonstrated in this letter that, in the context of semiclassical gravity, the last is indeed the right way to proceed with the problem and, thus, that the correct conclusions are those which follow from it. However, it will be also demonstrated that when one considers the theory based in the so-called reduced semiclassical Friedmann equation [12, 13], then it turns out that the right conclusions are obtained by following the first procedure. This will be discussed in detail in Sect. 4.

Specifically, we will consider in the paper the quantum effects due to massless, conformally coupled fields, as a typical example of a more general situation. This is a special, workable case where the quantum vacuum stress tensor—which depends on two regularization parameters, here called $\alpha > 0$ and $\beta < 0$—and the semiclassical Friedmann equation, can be calculated explicitly. Even if the equation is not integrable analytically, a detailed phase-space study can be performed on it. Such study shows that, for $-1 < \frac{\beta}{3\alpha} < 0$, the solutions without future singularities constitute a one-parameter family (a subclass of the two-parameter general solution), which evolves into the contracting Friedmann phase at late times, plus a particular solution, asymptotically converging towards the contracting de Sitter universe. The other solutions are singular and, for them, both the scale factor and the energy density anihilate in finite time. Actually, those solutions may bounce many times until, eventually, the scale factor falls in a contracting behavior which drives its value to zero in finite time. This clearly implies that, if one wants to match a solution which is in the expanding Friedmann phase at early times, with a solution without future singularities, instead of fine tuning the initial conditions what one needs to do is fine tune the two parameters $\alpha > 0$ and $\beta < 0$. Anyway, these non-singular solutions are unstable, since any small perturbation drives them to a singular behavior.

* E-mail: jaime.haro@upc.edu
† E-mail: jaume.amoros@upc.edu
‡ E-mail: elizalde@ieec.uab.es, elizalde@math.mit.edu

On the other hand, we will also prove that, when $-1 > \frac{\beta}{3\alpha}$, on top of the solutions without future singularities just described, the extra one corresponds to a universe bouncing infinitely often (an oscillating universe).

The paper is organized as follows. Next section is devoted to semiclassical gravity and the appearance of the future singularities. A phase-space study is carried out in Sect. 3. Sect. 4 is devoted to a study of the issue of the reduction of the order in the semiclassical Friedmann equation. Finally, in the last section we present the conclusions of the paper.

II. SEMICLASSICAL GRAVITY AND FUTURE SINGULARITIES

A. Big Rip singularity

In a flat Friedmann-Robertson-Walker metric the background equations are (the units used in this paper are $c = \hbar = M_p = 1$)

$$H^2 = \frac{\rho}{3} \quad \text{(Friedmann equation)}, \tag{1}$$

$$\dot{H} = -\frac{1}{2}(\rho + p) \quad \text{(acceleration equation)}, \tag{2}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \text{(conservation equation)}, \tag{3}$$

$$p = \omega\rho, \quad \omega < -1 \quad \text{(equation of state)}. \tag{4}$$

Deleting ρ from the Friedmann and the acceleration equations, and using the equation of state, one obtains $\dot{H} = -\frac{3}{2}(1 + \omega)H^2$, which can be integrated as

$$H_F(t) = \frac{2}{3(1+\omega)}\frac{1}{t - t_s}, \tag{5}$$

where $t_s \equiv -\frac{2}{3H_0(1+\omega)}$, being $H_0 = H(0)$ the initial condition. Assuming $H_0 > 0$, one has $t_s > 0$, that is, a future singularity [3].

B. Semiclassical gravity

For a massless, conformally coupled field, the anomalous trace is given by [5]

$$T_{vac} = \alpha\Box R - \frac{\beta}{2}G, \tag{6}$$

with R the scalar curvature and $G = -2\left(R_{\mu\nu}R^{\mu\nu} \; -\frac{1}{3}R^2\right)$ the Gauss-Bonnet curvature invariant, and where, to obtain this expression of G, we have used that the Weyl tensor vanishes in a FRW geometry.

The coefficients α and β, coming from dimensional regularization are [14]

$$\begin{aligned} \alpha &= \frac{1}{2880\pi^2}(N_0 + 6N_{1/2} + 12N_1) > 0, \\ \beta &= \frac{-1}{2880\pi^2}(N_0 + 11/2N_{1/2} + 62N_1) < 0, \end{aligned} \tag{7}$$

being N_0 the number of scalar fields, $N_{1/2}$ the number of four-component neutrinos, and N_1 the number of electromagnetic fields, respectively.

A remark is here in order. The constants α and β are fixed by the regularization process. For instance, dimensional regularization yields the expression (7) while point-splitting gives (see [15]) $\alpha = \frac{1}{2880\pi^2}(N_0 + 3N_{1/2} - 18N_1)$, $\beta = \frac{-1}{2880\pi^2}(N_0+11/2N_{1/2}+62N_1)$. As pointed out in [17], the value of α is arbitrary and it is influenced by the regularization method and also by the fields which are present in our universe; on the other hand, β is independent of the renormalization scheme and must be negative. Since the interesting case is $\alpha > 0$ (for $\alpha < 0$ there are no bouncing solutions [14]), we will here just consider values $\alpha > 0$ and $\beta < 0$.

In terms of the Hubble parameter, Eq. (6) is [15]

$$T_{vac} = 6\alpha(\dddot{H} + 12H^2\dot{H} + 7H\ddot{H} + 4\dot{H}^2) - 12\beta(H^4 + H^2\dot{H}). \tag{8}$$

With the trace anomaly $T_{vac} = \rho_{vac} - 3p_{vac}$ and inserting (8) into the conservation equation, $\dot{\rho}_{vac} + 3H(\rho_{vac} + p_{vac}) = 0$, the semiclassical vacuum energy density is is

$$\rho_{vac} = 6\alpha(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2) - 3\beta H^4 + Ca^{-4}, \tag{9}$$

being C a constant of integration which, in our case (flat FRW space-time), is equal to zero. This constant may be determined by direct calculation using regularization, or either may be obtained as follows. For a static space-time $a = a_0$ =const, Eq. (9) reduces to $\rho_{vac} = Ca_0^{-4}$, and the flat FRW space-time reduces to the Minkowski one, for which $\rho_{vac} = 0$. Thus, in the flat case $C = 0$. However, it is very interesting to note that, on the other hand, for a closed space-time one has $C = 6(\alpha - \beta)$ (see [15] for details) and, in this situation, the last term of (9) is the Casimir energy contribution, which is probably bound to have cosmological consequences (see, e.g., [16]).

From (9) we obtain the semiclassical Friedmann equation $H^2 = \frac{\rho + \rho_{vac}}{3}$.

In terms of the dimensionless variables $\bar{t} = H_+ t$, $\bar{H} = H/H_+$, $\bar{Y} = \dot{H}/H_+^2$, and $\bar{\rho} = \frac{\rho}{3H_+^2}$, with $H_+ = \sqrt{-1/\beta}$, the semiclassical Friedmann equation and the conservation equation can be written as an autonomous system, whose general solution is a two-parameter family:

$$\left\{ \begin{array}{rcl} \bar{H}' & = & \bar{Y}, \\ \bar{Y}' & = & \frac{1}{2\alpha\bar{H}}\left(-\beta\bar{H}^2 + \beta\bar{\rho} - 6\alpha\bar{H}^2\bar{Y} + \alpha\bar{Y}^2 + \beta\bar{H}^4\right), \\ \bar{\rho}' & = & -3\bar{H}(1+\omega)\bar{\rho}, \end{array} \right. \tag{10}$$

where $'$ denotes derivative with respect to $\bar{t}$ time. In absence of quantum corrections the energy density clearly diverges. No wonder one then finds future singular behaviors, of the form $\bar{\rho} = \bar{\rho}_0(\bar{t}_s - \bar{t})^\mu$, with $\mu < 0$ and $\bar{t}_s > \bar{t}$. Using the conservation equation, one gets $\bar{H} = \frac{\mu}{3(1+\omega)(\bar{t}_s - \bar{t})}$, and inserting these expressions into (10), and picking up the most singular terms, one obtains $\mu = -4$ and $\bar{\rho}_0 < 0$. But this is a contradiction, for the energy density is a positive quantity. The conclusion is that there are no future singularities in the region $\bar{H} > 0$. Note the difference with the Friedmann solution (5), which develops a future singularity when $\bar{H} > 0$.

This contrasts with the conclusions in [7, 11]. Specifically, in [11], starting from the classical Friedmann solution (5), the authors obtained the dark energy density

$$\rho_F = \frac{3H_F^2}{8\pi} = \frac{1}{6\pi(1+\omega)^2}\frac{1}{(t - t_s)^2}, \tag{11}$$

which, in terms of the scalar curvature, $R = 6(2H^2 + \dot{H}) = \frac{4(1-3\omega)}{3(1+\omega)^2}\frac{1}{(t-t_s)^2}$, can be written as $\rho_F = \frac{R}{8\pi(1-3\omega)}$. Inserting then the classical Friedmann solution into Eq. (9), for a scalar field ($\alpha = -\beta = \frac{1}{2880\pi^2}$), we get

$$\rho_{vac} = \frac{1}{34560\pi^2}\frac{27\omega^2 + 18\omega - 5}{(1-3\omega)^2}R^2. \tag{12}$$

After this, comparing Eqs. (11) and (12), it was argued that, for realistic values of ω (such as $\omega = -1.25$), when the scalar curvature is well below the Planck scale ($R < 1$), one had $\rho_{vac} \ll \rho_F$. Thus, the reached conclusion was that quantum effects could not significantly affect the universe expansion, up to the point where the spacetime curvature is of the order of the Planck scale and the semiclassical approximation breaks down. That is, semiclassical gravity would not soften, nor eventually avoid, the Big Rip singularity.

On the other hand, in [7], by inserting the classical Friedmann solution into (8) and (9), for scalar fields, the authors found the ratio of the vacuum pressure to the vacuum energy density to be given by $\omega_{vac} = \frac{p_{vac}}{\rho_{vac}} = 1 + 2\omega$, from where they calculated an effective equation of state with parameter $\omega_{eff} \equiv \frac{p + p_{vac}}{\rho + \rho_{vac}} = \omega + (1+\omega)\frac{\rho_{vac}}{\rho + \rho_{vac}} < \omega$. Which meant that the effect of the vacuum energy density of the quantized field would be to strengthen the accelerated expansion leading to a Big Rip singularity.

In contrast with those conclusions, we will next probe a rigorous way to analyze the problem, by carrying out a qualitative phase-space study, which can be checked numerically with good precision.

III. PHASE-SPACE STUDY

We shall use the following preliminary results. First, we look for future singular solutions, in the region $\bar{H} < 0$, with the form $\bar{H} = C/(\bar{t} - \bar{t}_s)$, when $\bar{t} \to \bar{t}_s$. Since $\bar{\rho}(t) \to 0$ in this region, one can insert this Hubble parameter into the semiclassical Friedmann equation, with $\bar{\rho} = 0$, to obtain $C_\pm = \frac{3\alpha}{\beta}\left(-1 \pm \sqrt{1 + \frac{\beta}{3\alpha}}\right)$ for $-1 \leq \frac{\beta}{3\alpha} < 0$. Proceeding in the same way as in [18], this result shows that there are two two-parameter families of solutions sharing this singular behavior at late times. This only happens if $-1 \leq \frac{\beta}{3\alpha} < 0$. When $-1 \geq \frac{\beta}{3\alpha}$, we will see below that the universe must cross the plane $\bar{H} = 0$ infinitely many times.

Second, the manifold $\bar{\rho} = 0$ is invariant; more precisely, the half plane $\bar{\rho} = 0$ with $\bar{H} > 0$ is a repeller, and the other one an attractor. Moreover, the system has two critical points $(\pm 1, 0, 0)$, and if one restricts the system to remain on the plane $\bar{\rho} = 0$, the point $(1, 0, 0)$ is then an attractor because it linearized system has eigenvalues $\lambda_\pm = -3/2\left(1 \pm \sqrt{1 + \frac{4\beta}{9\alpha}}\right)$ and $\lambda_3 = -3(1+\omega)$. At the other critical point the eigenvalues of the linearized system are $\lambda_\pm = 3/2\left(1 \pm \sqrt{1 + \frac{4\beta}{9\alpha}}\right)$ and $\lambda_3 = 3(1+\omega)$. Since $\mathcal{R}e(\lambda_\pm) > 0$ and $\bar{\rho} \equiv 0$ is an invariant manifold, all solutions in that semi-plane with $\bar{H} < 0$ escape from this critical point. Moreover since the eigenvector corresponding to the eigenvalue λ_3 is $\vec{v}_3 = (1, 3(1+\omega), -18\omega(1+\omega)\alpha/\beta + 2)$, there is a particular solution that goes asymptotically towards the de Sitter contracting phase following the direction of the vector $\vec{v}_3$ (see [18] for details).

Third, if one considers the Friedmann solution, in dimensionless variables,

$$\bar{Y}_F = -\frac{3}{2}(1+\omega)\bar{H}^2; \quad \bar{\rho}_F = \bar{H}^2, \tag{13}$$

then, as $|\bar{H}| \ll 1$, by linearizing the system (10) around the Friedmann solution, i.e., writting $\bar{Y} = \bar{Y}_F + \bar{Y}_{lin.}$ and $\bar{\rho} = \bar{\rho}_F + \bar{\rho}_{lin.}$, one obtains

$$\begin{pmatrix} \bar{Y}_{lin.} \\ \bar{\rho}_{lin.} \end{pmatrix}' = \frac{1}{\bar{H}} \begin{pmatrix} -\frac{2\beta}{9\alpha(1+\omega)^2} & -\frac{\beta}{3\alpha(1+\omega)\bar{H}^2} \\ \frac{4}{3(1+\omega)} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \bar{Y}_{lin.} \\ \bar{\rho}_{lin.} \end{pmatrix}, \tag{14}$$

and applying the WKB method [20], one gets the following one-parameter family of solutions that converges to the Friedmann one when $\bar{H} \to 0$

$$\begin{pmatrix} \bar{Y} \\ \bar{\rho} \end{pmatrix}_\pm \cong \begin{pmatrix} -\frac{3}{2}(1+\omega)\bar{H}^2 \\ \bar{H}^2 \end{pmatrix} + K \frac{e^{\pm \frac{2}{3(1+\omega)\bar{H}}\sqrt{-\frac{\beta}{\alpha}}}}{\sqrt{\bar{H}}} \begin{pmatrix} 1 \\ \mp 2\bar{H}\sqrt{-\frac{\alpha}{\beta}} \end{pmatrix} \quad \text{for} \pm \bar{H} > 0, \tag{15}$$

K being a free parameter.

As last issue, in order to qualitatively study the system, it is rather convenient, as in [19, 21], to perform the change of variable $\bar{p} \equiv \sqrt{|\bar{H}|}$. After what the semiclassical Friedmann equation becomes

$$\frac{d}{d\bar{t}}\left((\bar{p}')^2/2 + V(\bar{p})\right) = -3\epsilon p^2 (\bar{p}')^2 - \frac{3\beta\epsilon}{8\alpha}(1+\omega)\bar{\rho}, \tag{16}$$

where $V(\bar{p}) = \frac{\beta}{8\alpha}\left(\bar{p}^2(1 - \frac{1}{3}\bar{p}^4) + \frac{\bar{\rho}}{\bar{p}^2}\right)$, and $\epsilon \equiv \text{sign}\ (\bar{H})$. The potential V (Fig. 3 of Ref. [19]), has a unique zero, at $\bar{p}_0 = (3/2)^{1/4}\left(1 + \sqrt{1 + \frac{4}{3}\bar{\rho}}\right)^{1/4}$, and two critical points, at $\bar{p}_\pm = \left(\frac{1 \pm \sqrt{1 - 4\bar{\rho}}}{2}\right)^{1/4}$ ($\bar{p}_- < \bar{p}_+$). Thus, for $\bar{\rho} > 1/4$ there are no critical points, being the potential strictly increasing, from $-\infty$ to ∞. For $\bar{\rho} < 1/4$, the potential satisfies $V(0) = -\infty$ and $V(\infty) = \infty$, and has a relative maximum, at $\bar{p}_-$, and a relative minimum, at $\bar{p}_+$ (a hollow one). For very small values of $\bar{\rho}$, at $\bar{p}_-$ one has $\bar{H}^2 \cong \bar{\rho}$, that is, the system is close to the Friedmann phase and, at $\bar{p}_+$, one has $|\bar{H}| \cong 1$, that is, the system is near the de Sitter phase.

An interesting property of the system is that its energy, $E = (\bar{p}')^2/2 + V(\bar{p})$, changes sign when the trajectory crosses the plane $\bar{H} = 0$ (details are given in [18, 21]). Another important property is that the system is dissipative (resp. anti-dissipative) for $\bar{H} > 0$ (resp. $\bar{H} < 0$), what means that the system gains (resp. losses) energy when it returns to the region $\bar{H} < 0$ (resp. $\bar{H} > 0$). Being more specific, consider a trajectory that bounces many times, i.e., it crosses the plane $\bar{H} = 0$ repeatedly, and let E_0 be the energy of the system while in $\bar{H} < 0$ at a given time. Then, since the system is anti-dissipative in this region, its energy increases until the system bounces. Let $E_1 > E_0$ be the energy of the system when it crosses the plane $\bar{H} = 0$; at this moment the energy changes its sign and decreases, until the next bounce occurs, i.e., one has $-E_1 > -E_2$, where $-E_2$ is the energy at the next bouncing time. When the system enters the region $\bar{H} < 0$ its energy is $E_2 > E_1 > E_0$. This is to say, the system returns to the region $\bar{H} < 0$ having increased its energy. Similarly, one can show that the system losses energy when it returns back to the region $\bar{H} > 0$.

Taking all this into account, we can start our qualitative analysis, for $-1 < \frac{\beta}{3\alpha} < 0$. Since nowadays our universe is nearly to the expanding Friedmann phase driven by a phantom fluid, we assume that the system is close to this expanding phase at early times (a one-parameter family of solutions is in this phase, at early times), e.g., close to the point $\bar{p}_-$ with $\bar{H} > 0$. Subsequently, it leaves this expanding Friedmann state as it starts to roll down, either to the left or to the right. In the former case (Fig. 1), the universe rolls down to $\bar{p} = 0$ and enters a contracting phase $\bar{H} < 0$. Then, asymptotically, it can either reach one of the points $\bar{p}_-$ or $\bar{p}_+$ (the non-singular solutions at late time, i.e., the contracting de Sitter phase, and the one-parameter family in the contracting Friedmann phase), or keep on bouncing repeatedly, in order to acquire enough energy, in $\bar{H} < 0$, to arrive at $\bar{p} = \infty$ in finite time (the two-parameter singular solutions given by $\bar{H} = C_\pm/(\bar{t} - \bar{t}_s)$).

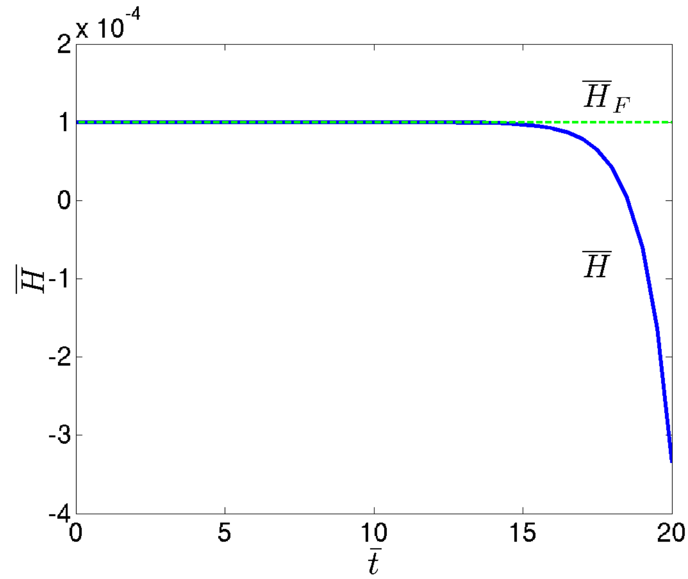

Figure 1: $\bar{H}(\bar{t})$ obtained by integration of Eqs. (10) with initial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and the classical Friedmann solution $\bar{H}_F(\bar{t}) = \frac{\bar{H}_0}{1+\frac{3(1+\omega)}{2}\bar{H}_0\bar{t}}$ in dimensionless variables ($\frac{\beta}{3\alpha} = -\frac{1}{3}, \omega = -1.25, \bar{H}_0 = 10^{-4}$). This plot clearly shows that, for small values of the scalar curvature, the solution of (10), which initially is in the expanding Friedmann phase, drastically differs from the classical Friedmann one.

If, on the contrary, the universe rolls down to the right (Fig. 2) it will approach the expanding de Sitter phase (the relative minimum $\bar{p}_+$). It will either pass by $\bar{p}_+$ and return back to the left, overcoming the maximum $\bar{p}_-$ and rolling down to $\bar{p} = 0$, or it will remain at $\bar{p}_+$ until $\bar{\rho}$ reaches $1/4$, at which moment the critical point will disappear and the potential will become an increasing function of $\bar{p}$, which means that the universe rolls down to $\bar{p} = 0$ and enters into a contracting phase $\bar{H} < 0$.

On the other hand, when $-1 > \frac{\beta}{3\alpha}$ the system bounces infinitely often (Fig. 3), since in that case the anti-dissipation effect is not large enough—as compared to the potential force—so that, when the universe is in the contracting region, it is bound to necessarily return to the expanding one, and vice versa.

To conclude, when the parameters satisfy $-1 < \frac{\beta}{3\alpha} < 0$ one must very finely tune the initial conditions and the parameters α and β in order to obtain non-singular solutions which match these late time non-singular behaviors with the expanding Friedmann stage at early times. This is so, because those families of solutions do not constitute a general solution (a two-parameter family). Moreover, the non-singular solutions are unstable at late times, in the sense that a small perturbation of their initial conditions leads them to become singular. Note that, our analytical study do not allow us to say any thing else about the dynamics of the system. In particular, its impossible to determine analytically the relationship between the parameter and , and the evolution towards the non-singular solutions or towards the singular ones. Only a very detailed numerical study could give us some information about this relation, but this deserves future investigations.

On the other hand, when $-1 > \frac{\beta}{3\alpha}$, apart from the non-singular solutions described above, the other solutions corre-

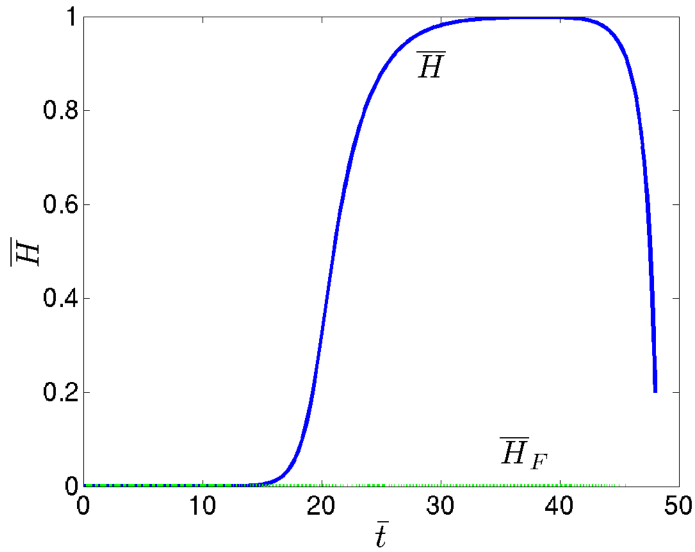

Figure 2: Same as in Fig. 1 but with initial conditions $(\bar{H}_0, -\frac{3}{2}(0.75+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and parameters $(\frac{\beta}{3\alpha} = -\frac{1}{3}, \omega = -1.25, \bar{H}_0 = 10^{-4})$. The universe rolls down to the right, passes the expanding de Sitter phase ($\bar{p}_+$), returns to the left, overcoming the expanding Friedmann phase ($\bar{p}_-$), and finally enters into a contracting phase.

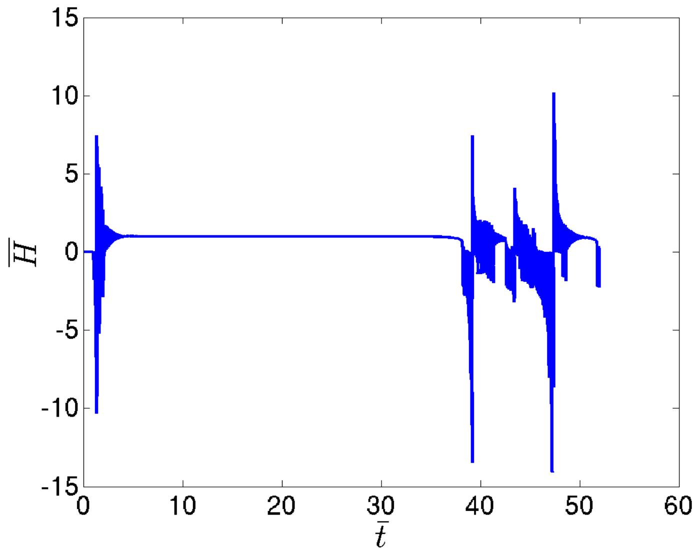

Figure 3: Same as in previous figures, with initial conditions $(\bar{H}_0, -\frac{3}{2}(1+\omega)\bar{H}_0^2, \bar{H}_0^2)$ and parameters $(\frac{\beta}{3\alpha} = -\frac{10000}{3}, \omega = -1.25, \bar{H}_0 = 10^{-6})$. The universe starts oscillating around $\bar{H} = 0$, then enters into a expanding de Sitter phase until the energy density reaches $1/4$, at which point the universe leaves this phase, rolling down to the contracting phase, and from there it starts to oscillate once again around $\bar{H} = 0$.

spond to a universe bouncing infinitely many times.

IV. REDUCTION OF THE ORDER IN THE SEMICLASSICAL FRIEDMANN EQUATION

As we have seen, solutions of the semiclassical Friedmann equation differ drastically from the corresponding classical ones. In this respect, some authors argue that not all of the new solutions, allowed by the semiclassical corrections, fall within the perturbative framework and, thus, they conclude that the way to obtain a self-consistent approach is to exclude all these non-perturbative solutions [12, 13, 22]. The procedure proposed to eliminate these solutions goes as follows. The semiclassical Friedmann equation is a second order-derivative equation on H whereas the classical one is a zero order equation; thus, the idea is to reduce the order of the semiclassical Friedmann equation to a zero order equation—the same of the classical equation—in such a way that the new solutions turn out to be small perturbations of the classical ones. The problem of this prescription is that it excludes many interesting possibilities, in particular all the solutions which correspond to inflationary models like the Starobinsky one [23].

Other authors consider a less drastic alternative and only disregard solutions which grow without bound at late times [24–26], as for instance the runaway solutions of the Lorentz-Dirac equation of motion when the Abraham-Lorenz-Dirac radiation reaction force is taken into account [27].

Here we follow the approach proposed by Parker and Simon [12, 13]. The leading idea is to obtain the derivatives of H from the Friedmann and conservation equation, and to insert the obtained expressions into the semiclassical equation, which gives the following reduced Friedmann equation

$$H^2 = \frac{\rho}{3} + \frac{H^4}{h^2}, \tag{17}$$

where $h \equiv \left[\frac{9}{4}\alpha(1+\omega)(3\omega-1)-\beta\right]^{-1/2} > 0$. From (17) one deduces that $-h \leq H \leq h$ because the energy density must be positive. Now, taking the derivative of (17) and using the conservation equation, one obtains the following first order equation, which can be explicitly integrated:

$$\dot{H} = -\frac{3}{2}(1+\omega)H^2\left(\frac{h^2-H^2}{h^2-2H^2}\right). \tag{18}$$

The solution of Eq. (18) is given by

$$-1/H(t) + \frac{2}{h}\ln\left|\frac{H(t)-h}{H_0-h}\frac{H_0+h}{H(t)+h}\right| = -\frac{3}{2}(1+\omega)(t-t_s), \tag{19}$$

where $t_s \equiv -\frac{2}{3H_0(1+\omega)}$, $H_0 = H(0)$ being the initial condition.

We can now argue that, if the initial condition belongs in $(0, h/\sqrt{2})$ then, when $t \to -\infty$ on has $H \to 0$, and when $H \to h/\sqrt{2}$ at $\bar{t}_s \equiv t_s + \frac{2}{3(1+\omega)}\left(\sqrt{2}/h - \frac{2}{h}\ln\left|\frac{1-\sqrt{2}}{H_0-h}\frac{H_0+h}{1+\sqrt{2}}\right|\right) < t_s$ it immediately follows from (18) that $\dot{H}(\bar{t}_s) = +\infty$, that is, these solutions are singular (the scalar curvature, $R \equiv 6(2H^2+\dot{H})$, diverges at $\bar{t}_s$). If H_0 belongs in $(h/\sqrt{2}, h)$ when $t \to -\infty$ one has $H \to h$; however, at $\bar{t}_s$, $\dot{H}$ diverges.

From this last result the difference between the semiclassical Friedmann equation and the reduced Friedmann one (equation (17)) becomes apparent. In fact, we have shown before that the solutions of the semiclassical Friedmann equation do not develop singularities in the region $H > 0$, since actually they bounce and enter into the contracting region. However, the last result above shows that the solutions of the reduced Friedmann equation, like the solutions of the classical one, do develop future singularities in the region $H > 0$.

As last remark, note that if one inserts on the right hand side of the reduced Friedmann equation (17), the Friedmann equation $H^2 = \rho/3$, one will obtain a different reduced equation, namely

$$H^2 = \frac{\rho}{3} + \frac{\rho^2}{9h^2} \Longleftrightarrow \frac{(\rho+3h^2/2)^2}{9h^4/4} - \frac{H^2}{h^2/4} = 1. \tag{20}$$

Taking now the derivative of this equation, we get

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\rho}{6}(1+\omega) - \frac{\rho^2}{9h^2}(2+3\omega) > -\frac{\rho}{6}(1+\omega), \tag{21}$$

which means that, in the corresponding theory based on this equation, the effect of the vacuum energy density is to strengthen the accelerated expansion that leads to the Big Rip singularity, and thus, in this context, the results obtained in [7, 8] are right. In fact, on the plane (H, ρ), the universe evolves following the hyperbola given in Eq. (20), whereas in classical cosmology it evolves according to the parabola $H^2 = \rho/3$.

V. CONCLUSIONS

As a general conclusion of our work, it is important to stress that the results here obtained by using the reduced Friedmann equation agree with those of [7, 11]. This is because, essentially, both ways to analyze the issue of the avoidance of future singularities are the same. They rely in treating the vacuum quantum effects as small perturbations and, thus, the introduction of the classical Friedmann solution in the energy density of the quantum field is allowed. On the other hand, if one starts from the semiclassical Friedmann equation, then quantum vacuum effects cannot be treated, in general, as an small perturbation and they can change drastically the future expansion of the universe.

To finish, what one can conclude from our work is that the results obtained in [5, 6, 14, 18, 19, 21] are only right in the context of semiclassical gravity, whereas the conclusions of [7, 8, 11] only holds if one considers the semiclassical theory based on the reduced semiclassical Friedmann equation.

Acknowledgments. This investigation has been supported in part by MICINN (Spain), projects MTM2008-06349-C03-01, MTM2009-14163-C02-02, FIS2006-02842 and FIS2010-15640, by CPAN Consolider Ingenio Project, and by AGAUR (Generalitat de Catalunya), contracts 2009SGR 345, 994 and 1284.

[1] S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).
[2] A.G. Riess et al., Astron. J. **116**, 1009 (1999).
[3] R.R. Caldwell, M. Kamionkowski and N.N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003).
[4] J.D. Barrow, Class. Quantum. Grav. **21**, L79 (2004); J.D. Barrow, Class. Quantum Grav. **21**, 5619 (2004).
[5] S. Nojiri, S. Odintsov and S. Tsujikawa, Phys. Rev. **D71**, 063005 (2005).
[6] S. Nojiri and S. Odintsov, Phys. Lett. **B595**, 1 (2004).
[7] H. Calderón and W.A. Hiscock, Class. Quantum Grav. **22**, L23 (2005).
[8] H. Calderón, Phys. Rev. **D78**, 044041 (2008).
[9] J.D. Barrow, A.B. Batista, J.C. Fabris and S. Houndjo, Phys. Rev. **D78**, 123508 (2008).
[10] S.K. Srivastava, Gen. Relativ. Gravit. **39**, 241 (2007).
[11] J.D. Bates and P.R. Anderson, Phys. Rev. **D82**, 024018 (2010).
[12] J.Z. Simon, Phys. Rev. **D45**, 1953 (1992); **D41**, 3720 (1990).
[13] L. Parker and J.Z. Simon, Phys. Rev. **D47**, 1339 (1993).
[14] M.V. Fischetti, J.B. Hartle and B.L. Hu, Phys. Rev. **D20**, 1757 (1979).
[15] P.C.W. Davies, Phys. Lett. **B68**, 402 (1977).
[16] E. Elizalde, S. Nojiri, S.D. Odintsov and S. Ogushi, Phys. Rev. **D67**, 063515 (2003); E. Elizalde, S. Nojiri and S. Odintsov, Phys. Rev. **D70**, 043539 (2004); E. Elizalde, S. Nojiri, S.D. Odintsov and P. Wang, Phys. Rev. **D71**, 103504 (2005); E. Elizalde, J. Phys. **A39**, 6299 (2006).
[17] R.M. Wald, Phys. Rev. **D17**, 1477 (1978).
[18] J. Haro, arXiv (gr-qc): 1011.4772 (2010).
[19] S. Wada, Phys. Rev. **D31**, 2470 (1985).
[20] M. Fedoriouk, *Méthodes asymptotiques pour les équations différentielles ordinaires linéaires* (Editions Mir, France, 1987).
[21] T. Azuma and S. Wada, Prog. Theor. Phys. **75**, 845 (1986).
[22] E.E. Flanagan and R.M. Wald, Phys. Rev. **D54**, 6233 (1996).
[23] A.A. Starobinski, Phys. Lett. **B91**, 99 (1980).
[24] B.L. Hu, A. Roura and E. Verdaguer, IJTP **43**, 749 (2004).
[25] B.L. Hu, A. Roura and E. Verdaguer, Phys. Rev. **D70**, 044002 (2004).
[26] S.W. Hawking, T. Hertog and H.S. Reall, Phys. Rev. **D63**, 083504 (2001).
[27] J.D. Jackson, *Classical Electrodynamics*, Wiley, New York (1999).